Universality of dynamic processes using Drinfel'd twisters

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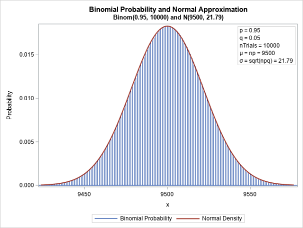
However, the alternative text may be tedious and the artifacts are untagged.

Many probabilistic theorems are motivated by *universality*. The classical example is the central limit theorem, which states that for independent, identically distributed random variables $\{X_n\}_{n\geq 0}$ with mean μ and variance σ^2,

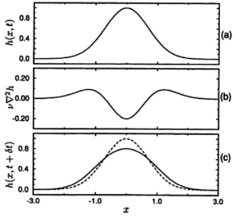
$$\frac{X_1 + \cdots X_n - \mu n}{\sigma n^{1/2}} \xrightarrow{d} N(0,1),$$

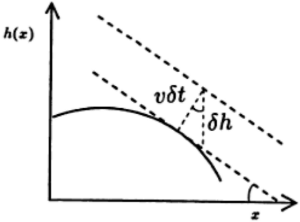
where $N(0,1)$ is the standard Normal distribution introduced by [Adrain, 1808] and [Gauss, 1809].



The KPZ equation was introduced by physicists Kardar, Parisi, Zhang '86 to model surface growth models undergoing *relaxation* and *lateral growth*. Non–rigorously, the height function $h(x,t)$ evolves as

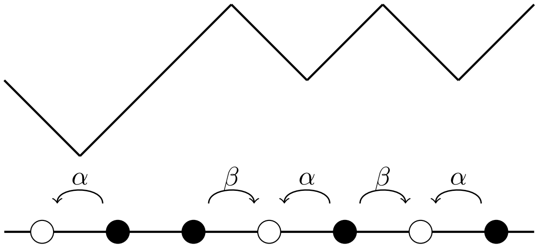
$$\frac{\partial h(x,t)}{\partial t} = \underbrace{\nu \nabla^2 h}_{\text{relaxation}} + \underbrace{\frac{\lambda}{2}(\nabla h)^2}_{\text{lateral growth}} + \underbrace{\eta(x,t)}_{\text{space–time white noise}} .$$





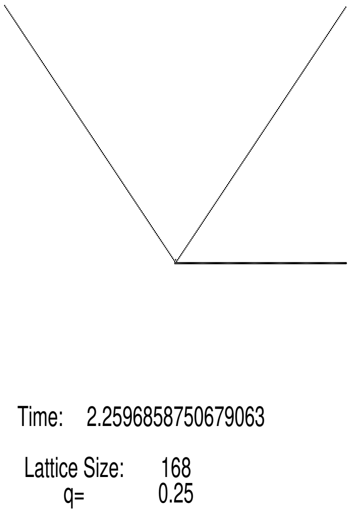
The KPZ equation is ill–defined, due to the non–linear term. Nonetheless, using renormalization group arguments from physics, Kardar–Parisi–Zhang predicted $t^{1/3}$ scaling limit, rather than $t^{1/2}$.

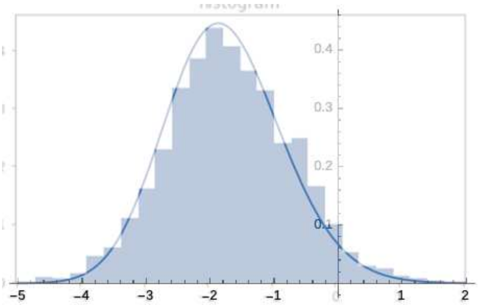
Consider a discretization of the KPZ equation, called the asymmetric simple exclusion process (ASEP) on a one–dimensional lattice, introduced by MacDonald–Gibbs–Pipkin ('68), Spitzer ('70):



Let $q = \sqrt{\beta/\alpha} \neq 1$ denote the asymmetry parameter. In the symmetric case ($q = 1$) the model discretizes the Edwards–Wilkinson equation, which has no non–linear term.

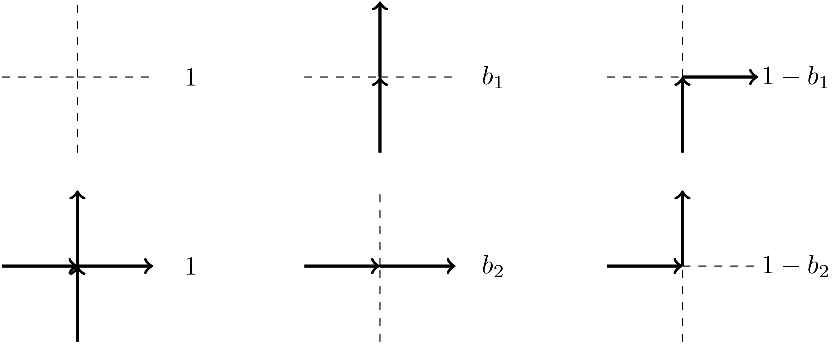
Consider step initial conditions. The animation below shows a growing parabola with random fluctuations.



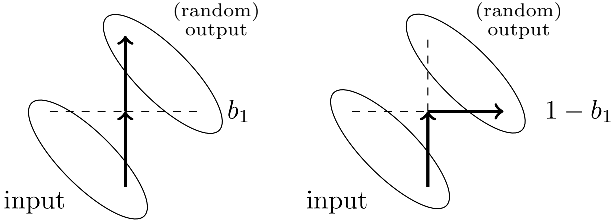


Johansson '03 ($q = 0$) and Tracy and Widom '08 ($q \neq 1$) proved that the fluctuations have $t^{1/3}$ scaling exponent with the so–called "Tracy–Widom" distribution, which first occurred in fluctuations of the largest eigenvalue of random matrices. Mathematically, it is not immediately obvious how eigenvalues of random matrices and interacting particle systems would be related to each other. Simulations were done with the aid of the Texas A&M High Performance Resource Center.

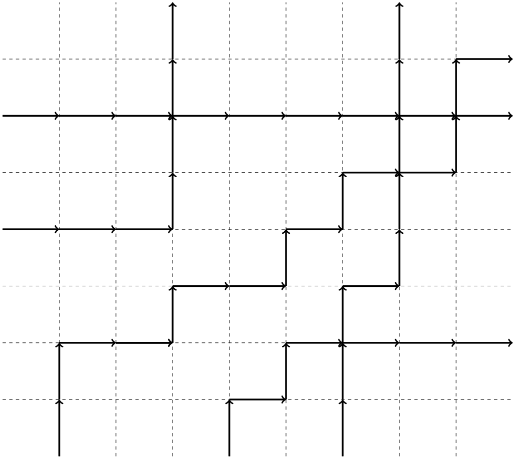
ASEP is also a degeneration of the <u>stochastic six vertex model</u>, introduced by Gwa–Spohn ('92). The parameters satisfy $0 \leq b_1, b_2 \leq 1$. The asymmetry parameter is now $q = \sqrt{b_1/b_2}$. The general six vertex model goes back to Pauling ('35), Slater ('41).



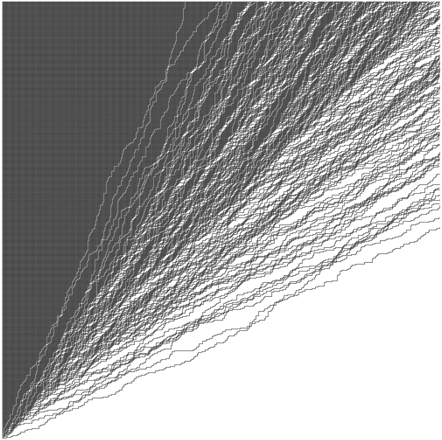
The probability measure can be defined by Markov update:



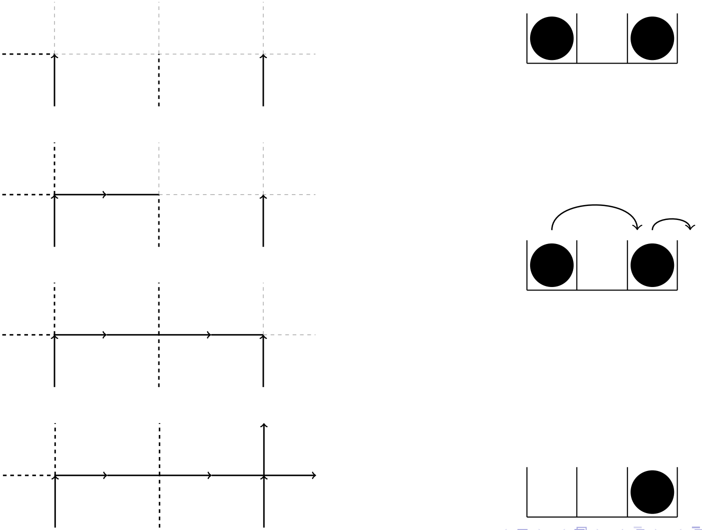
The weight of a configuration is the product of the weights at each vertex. This can be normalized to give a probability measure on configurations.



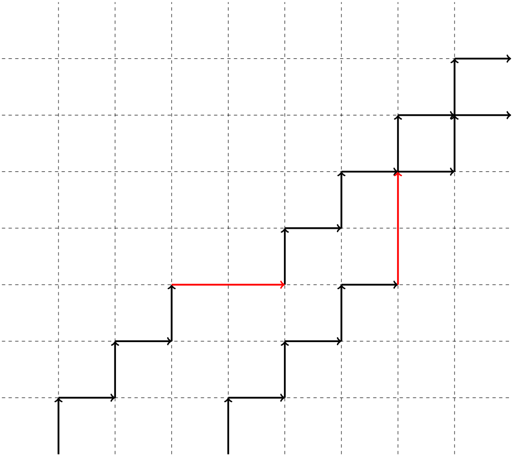
A simulation of the stochastic six–vertex model (courtesy of Leonid Petrov). The analog of step initial conditions is when arrows come in from the left and no arrows come in from the bottom.

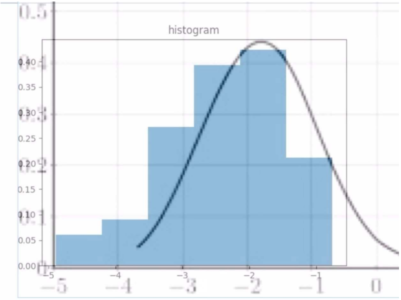


The vertex model can be viewed as a discrete–time particle system (on the infinite line or with *open* boundary conditions):



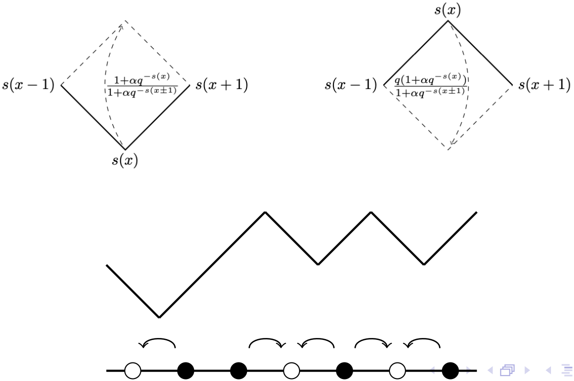
There is a degeneration from the stochastic six vertex model to ASEP when $b_1 = \epsilon, b_2 = q^2\epsilon$ as $\epsilon \to 0$.

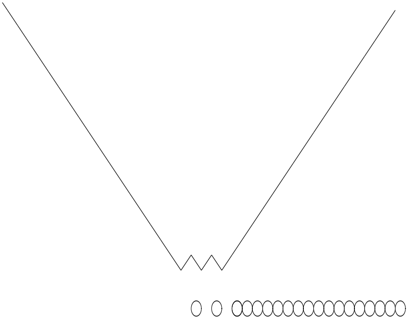




Borodin–Corwin–Gorin ’14 proved that the stochastic six vertex model also has Tracy–Widom fluctuations. The above histogram shows the height function at $(400000, 4000000)$ when $b_1 = 0.75, b_2 = 0.65$. These 48 simulations were done with the aid of the Texas A&M High Performance Resource Center.

In dynamic ASEP, there is a "dynamic" parameter $\alpha \in (0, \infty)$ in addition to the asymmetry parameter q. The dynamic parameter α interpolates between an ASEP with asymmetry to the left and an ASEP with asymmetry to the right. An equivalent interpretation is that the asymmetry parameter is "dynamically" changes from q to q^{-1} as the height function increases.





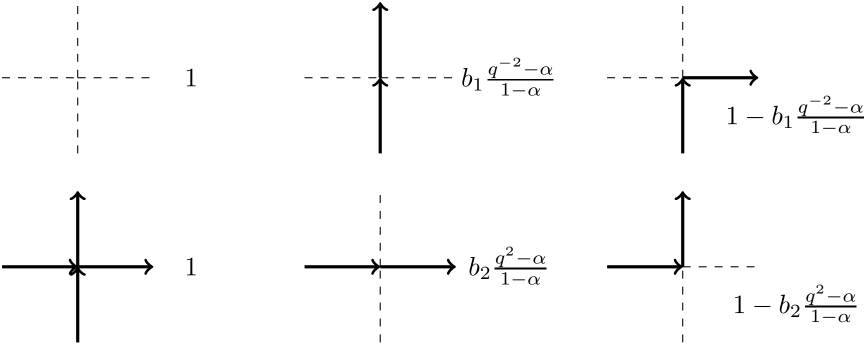
Animation showing a randomly evolving piecewise function.
It seems to not be changing very much from its initial condition.

Time: 1.0732568532479663

q = 0.5

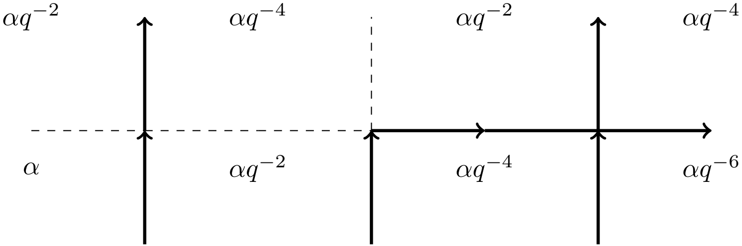
alpha = 0.2

The dynamic stochastic six vertex model has weights recall $q=\sqrt{b_1/b_2}$):



As with the dynamic ASEP, the dynamic parameter α interpolates between a non–dynamic model with parameters q and q^{-1}.

Alternatively, the asymmetry parameter changes over time:



Borodin '17 and Aggrawal '17 proved that for certain **symmetric** dynamic models, the height function has $t^{1/4}$ scaling to a non–deterministic limit. Corwin–Ghosal–Matetski '19 also studied a stochastic PDE limit. However, there were no conjectures for asymmetric dynamic models.

Theorem (K-Zhou '23)

If the dynamic parameter α depends on t in such a way that $\liminf \alpha(t)/t > 0$, then the dynamic stochastic six vertex model either has Tracy–Widom asymptotics with $t^{1/3}$ scaling exponent; otherwise it has exponentially vanishing fluctuations.

The "exponentially vanishing fluctuations" occur in the stochastic six vertex model with $q > 1$.

Theorem (K-Zhou, '24+)

If the dynamic parameter α depends on t in such a way that $\liminf \alpha(t)/t > 0$, then the dynamic ASEP either has Tracy–Widom asymptotics with $t^{1/3}$ scaling exponent; otherwise it has finitely bounded fluctuations.

The "finite fluctuations" occur in ASEP with $q > 1$.
In words, the asymptotics of the dynamic models with asymmetry parameter q have the same asymptotics as

- the non–dynamic model with asymmetry parameter q; or
- the non–dynamic model with asymmetry parameter q^{-1}.

The proof of the theorem can be broken down into a few steps:

- First, use the underlying algebraic structure to find an *orthogonal polynomial* duality function.
- Next, use the duality to reduce calculations of nth moments to the n–particle system.

Markov duality has a wide range of probabilistic applications; however, "finding dual processes is something of a black art." Etheridge ('06), Jansen–Kurt ('12).

Markov duality allows us to calculate certain expected values of the original Markov process in terms of a simple, "dual" process.

Suppose η_t and ζ_t are Markov processes with state spaces X and Y respectively, and let $D(\eta, \zeta)$ be a bounded measurable function on $X \times Y$. The processes η_t and ζ_t are said to be *dual* to one another with respect to D if

$$\mathbb{E}_\eta D(\eta_t, \zeta) = \mathbb{E}_\zeta D(\eta, \zeta_t) \text{ for all } t \geq 0.$$

An equivalent definition of duality (on discrete state spaces): If the generators L_X and L_Y are viewed as $X \times X$ and $Y \times Y$ matrices respectively, and D is viewed as a $X \times Y$ matrix, then

$$L_X D = D L_Y^T.$$

Here, the superscript T denotes transposition.

Orthogonal duality functions allow expectations to be expressed in terms of the duality function. Suppose there is a set of duality functions $D(x,y)$ such that $\{D(x,y)\}_{y\in Y}$ is an orthonormal basis of the Hilbert space $L^2(X,\nu)$, where ν is some measure on X. Then any function $h \in L^2(X,\nu)$ can be expressed in terms of

$$h(x) = \sum_{y\in Y} c_h(y) D(x,y)$$

where

$$c_h(y) = \sum_{x\in X} \nu(x) D(x,y).$$

Theorem

K-Zhou, '23 The dynamic stochastic six vertex model is dual to a (non—dynamic) space reversed stochastic higher spin vertex model. The duality functions are nested products of hypergeometric ${}_3\varphi_2$ *functions, evaluated at the height functions.*

The dynamic ASEP is dual to a (non–dynamic) space reversed ASEP with respect to the same duality function.

The duality functions are orthogonal respect to certain "blocking" measures ν *on the state space.*

Let $Y = \{y_1, \ldots, y_n\}$ denote the occupied particles in the dual process, and $\mu = (\mu_y)_{y\in\mathbf{Z}}$, where $\mu_y \in \{0,1\}$ is the number of particles in the configuration μ. Then, in a certain degeneration, the duality function takes the form

$$D(\mu, Y) = \left[\prod_{i=1}^{n}\left(1_{\{\mu_{y_i}=0\}}\alpha q^{2(h(y_i-1))} + 1_{\{\mu_{y_i}=1\}}q^{-2(y_i-1+h(y_i-1))}\right)\right]$$

Intuition: depending on the value of α, only the first term or the second term in the product will contribute asymptotically, corresponding to either q or q^{-1} in the non–dynamic case K. '16. The duality function then comes n–th q–moments, and exact formulas for the n–particle system yield asymptotics Borodin–Corwin–Sasamoto '12.

Idea behind algebraic understanding: associate the state space with a representation of an algebra, and interpret generator and duality through the action of algebra.

We will see how to derive the duality relation $L_{\mathrm{X}} D = D L_{\mathrm{Y}}^T$ algebraically, as well as how it generalizes.

Start by considering two lattice sites (with closed boundary conditions). There are four particle configurations, which can be identified with $\mathbb{C}^2 \otimes \mathbb{C}^2$ with basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\mathbb{C}^2$ is a representation of the Lie algebra $\mathfrak{sl}_2$ of traceless 2×2 matrices, which has basis

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{creation operator}$$

$$f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{annihilation operator}$$

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{number operator}$$

The action is given by explicit multiplication: for example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The Lie algebra $\mathfrak{sl}_2$ also acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$ through the co–product:

$$\Delta(e) = 1 \otimes e + e \otimes 1,$$
$$\Delta(f) = 1 \otimes f + f \otimes 1,$$
$$\Delta(h) = 1 \otimes h + h \otimes 1.$$

Any Lie algebra $\mathfrak{g}$ can be embedded in its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. For $\mathcal{U}(\mathfrak{sl}_2)$, the generators are e, f, h with relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Lifting to $\mathcal{U}(\mathfrak{g})$ allows us to multiply generators; for example, $e^2 \neq 0$. The co–product is required to be a homomorphism.

The asymmetry occurs through quantization. The Drinfeld–Jimbo ('86) quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ has q–deformed generators, relations and co–product. In particular, the co–product is no longer symmetric:

$$\begin{aligned}\Delta(e) &= q^h \otimes e + e \otimes 1,\\ \Delta(f) &= 1 \otimes f + f \otimes q^{-h},\\ \Delta(h) &= 1 \otimes h + h \otimes 1.\end{aligned}$$

The relations are

$$q^h e = q^2 e q^h, \quad f q^h = q^2 q^h f, \quad [e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}.$$

When $q \to 1$, we recover the usual $\mathcal{U}(\mathfrak{sl}_2)$.

Side remark: Other semi–simple Lie algebras can similarly be quantized, obtaining other duality results.

- Using $\mathcal{U}_q(\mathfrak{sl}_{n+1})$, so–called "multi–species"', there are duality results of Belitsky–Schütz '15, K. '15, K.'16, Belitsky–Schütz '16, K. '17.
- Other examples are $\mathfrak{sp}_4$ K. '15 and $\mathcal{U}_q(\mathfrak{so}_{2n})$ K–Landry–Lin–Park–Zhou '20, Blyschack–Burke–K–Li–Ustilovsky–Zhou '22, Rohr–Sellakumaran–Yin '23. One obtains a two–species asymmetric process with Markov duality.

For **asymmetric models** with open boundary conditions, there were no duality results until K. 19, K. 20, K. 21. These papers use other algebraic methods, such as the Schur–Weyl duality with Hecke algebras (Benkart–Witherspoon '01 or Bao–Wang–Watanabe '16).

Although the co–product is not symmetric, it is "almost" symmetric in the sense that there exists $R \in \mathcal{U}_q(\mathfrak{sl}_2)\hat{\otimes}\mathcal{U}_q(\mathfrak{sl}_2)$ such that

$$R\Delta(u) = \Delta'(u)R,$$

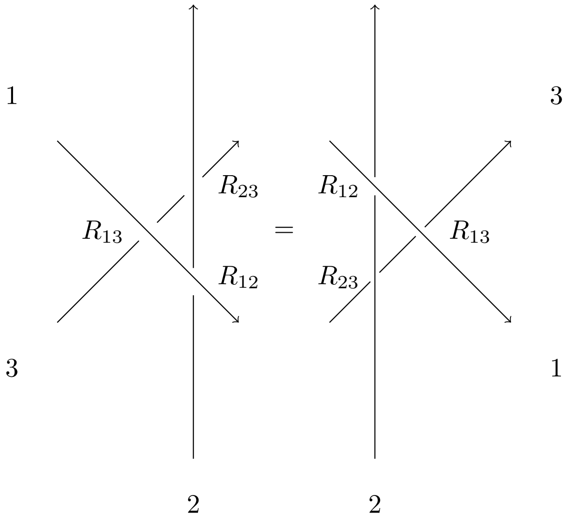
where $\Delta'(u)$ is the reversed co–product, defined by $\Delta' = P \circ \Delta$ with $P(u \otimes v) = v \otimes u$.
In more precise terminology, $\mathcal{U}_q(\mathfrak{sl}_2)$ is an "almost–cocommutative Hopf algebra." Furthermore, it is a quasi–triangular Hopf algebra, which implies that R satisfies the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Here, $R_{ij} = \phi_{ij}(R)$ where $\phi_{ij} : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ is defined by

$$\phi_{12}(u\otimes v) = u\otimes v\otimes 1, \qquad \phi_{13}(u\otimes v) = u\otimes 1\otimes v, \qquad \phi_{23}(u\otimes v) = 1\otimes u\otimes v.$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$



In addition to quantization, there is another generalization to *affine* Lie algebras. Given a finite–dimensional simple Lie algebra $\mathfrak{g}$, then (as an infinite–dimensional vector space)

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c.$$

Because of the additional term t, there is now a family of two–dimensional representations $\mathbb{C}^2(z)$ of $\widehat{\mathfrak{sl}_2}$, defined by letting t act as multiplication by the complex number z. The spectral–dependent version of the Yang–Baxter equation is

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(w),$$

Recall that H is the "number operator." The *dynamical Yang–Baxter equation* is

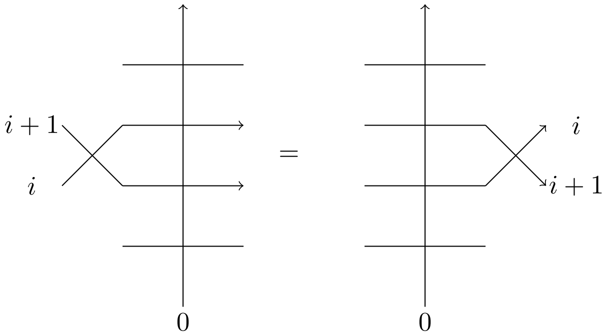
$$R_{12}(z,\alpha q^{-2H_3})R_{13}(zw,\alpha)R_{23}(w,\alpha q^{-2H_1}) = R_{23}(w,\alpha)R_{13}(zw,\alpha q^{-2H_2})R_{12}(z,\alpha),$$

where $H_1 = H \otimes 1 \otimes 1, H_2 = 1 \otimes H \otimes 1, H_3 = 1 \otimes 1 \otimes H$. An equivalent formulation is

$$R_{0,i+1}(z,\alpha - q^{2H_i})R_{0,i}(zw,\alpha)\check{R}_{i,i+1}(w,\alpha q^{-2\eta H_0}) = \check{R}_{i,i+1}(w,\alpha)R_{0,i+1}(zw,\alpha q^{-2\eta H_i})R_{0,i}(z,\alpha).$$

where $\check{R} = P \circ R$ (recall $P(u \otimes v) = v \otimes u$).

$$R_{0,i+1}(z,\alpha q^{-2H_i})R_{0,i}(zw,\alpha)\check{R}_{i,i+1}(w,\alpha q^{-2\eta H_0}) \\ = \check{R}_{i,i+1}(w,\alpha)R_{0,i+1}(zw,\alpha q^{-2\eta H_i})R_{0,i}(z,\alpha).$$



$$L \quad D \quad = \quad D \quad L$$

In the dYBE,

$$\underbrace{\check{R}_{i,i+1}(w,\alpha)}_{L}\underbrace{R_{0,i+1}(zw,\alpha q^{-2H_i})R_{0,i}(z,\alpha)}_{D} = \underbrace{R_{0,i+1}(z,\alpha q^{-2H_i})R_{0,i}(zw,\alpha)}_{D}\underbrace{\check{R}_{i,i+1}(w,\alpha q^{-2\eta H_0})}_{L}$$

the right–hand–side has H_0 equal to the ∞, since H_0 is the number operator equal to the difference in the number of particles between the original process and the dual process. This explains the duality between the dynamic model and the non–dynamic model.
The dynamic parameter only occurs in the duality function, which asymptotically will be the same duality function as the non–dynamic case.

Solutions to the dynamical Yang–Baxter equations can be constructed from quasi–triangular quasi–Hopf algebras. If the twister $F(\alpha)$ satisfies the shifted co–cycle condition

$$F_{12}(\alpha)(\Delta \otimes \mathrm{id})F(\alpha) = F_{23}(\alpha - 2\eta H_1)(\mathrm{id} \otimes \Delta)F(\alpha),$$

then the twisted R–matrix

$$R(z,\alpha) = F_{21}(\alpha)\,(R(z))^{-1}\,F_{12}^{-1}(\alpha)$$

satisfies the dynamical Yang-Baxter Equation.

The Drinfel'd Jimbo quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ has a Drinfel'd Twister $F(\alpha) \in \mathcal{U}_q(\mathfrak{sl}_2)\hat{\otimes}\mathcal{U}_q(\mathfrak{sl}_2)$. Additional properties of $\mathcal{U}_q(\mathfrak{sl}_2)$ can be used to relate L to L^T.

Theorem (K-Zhou, '23)

If the twisted R–matrix of a quasi–triangular quasi–Hopf algebra satisfies the relations (in addition to dynamic Yang–Baxter equation):

$$(F_{21}(\alpha)^T)^{-1}\Pi^{\otimes 2}F_{12}(\alpha)^{-1}PR(z,\alpha) = PR(z,\alpha)^T(F_{21}(\alpha)^T)^{-1}\Pi^{\otimes 2}F_{12}(\alpha)^{-1},$$
$$G_{i,i+1}^{-1}(\alpha)\check{R}_{i,i+1}(z,\alpha)G_{i,i+1}(\alpha) = \check{S}_{i,i+1}(z,\alpha)$$

where $S_{i,i+1}(z,\alpha)$ is a stochastic matrix and G is some diagonal matrix.
Further, we assume there is a diagonal matrix C and an involution Π such that

$$\Pi G(-\mathbf{i}\infty)\Pi = CPG(-\mathbf{i}\infty)P$$
$$\Pi^{\otimes 2}S(z,\alpha)\Pi^{\otimes 2} = S(z^{-1},\alpha).$$

Then there is an intertwining relationship $SD = DS^T$ which defines duality.

These relations are satisfied due to the algebraic structures of $\mathcal{U}_q(\mathfrak{sl}_2)$. For example,

$$G_{i,i+1}^{-1}(\alpha)\check{R}_{i,i+1}(z,\alpha)G_{i,i+1}(\alpha) = \check{S}_{i,i+1}(z,\alpha)$$

follows from the R–matrix being an element of $\mathcal{U}_q^{\geq 0}(\mathfrak{sl}_2) \otimes \mathcal{U}_q^{\leq 0}(\mathfrak{sl}_2)$ and

$$\Pi^{\otimes 2} S(z,\alpha)\Pi^{\otimes 2} = S(z^{-1},\alpha)$$

follows from a non–trivial automorphism of the Dynkin diagram.

How do we know the dualities will be orthogonal? The orthogonality of the duality functions comes from the $*$–Hopf algebra structure of the quantum group. This was first demonstrated in

Theorem (Franceschini, K., Zhou, '22)

The "n-species" ASEP (and a partial exclusion generalization) has orthogonal polynomial dualities, which are the q–Krawthcouk polynomials.

This model can be constructed using quantum group $\mathcal{U}_q(\mathfrak{gl}_{n+1})$, which has a *–Hopf algebra structure when q is a nonzero real number. Calculations were done using the q–Baker–Campbell–Hausdorff formula.

This model actually does *not* satisfy the Yang–Baxter equation: the orthogonal polynomial dualities were constructed using unitary symmetries of the quantum group.

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