TAMU 2013 Freshman-Sophomore Math Contest

Both Versions

While the name of the contest is traditional, the actual eligibility rules are that first year students take the freshman contest, and second year students take the sophomore contest. That way, students who have accummulated enough credit hours in their first or second year to have standing as sophomores, or juniors, are not promoted out of eligibility.

The first page contains problems built around Calculus I and II for both freshmen and sophomores. The second pages are pitched to content unique to Calculus III and/or Differential Equations, in the case of the sophomore contest.

In all cases, solutions should be written out and should include reasoning behind the steps when reasons beyond routine calculation are involved. No tables, calculators, or computers, and no devices for communication with the outside world, are allowed. You're on your own.

1. Find
$$\lim_{x\to\infty} \frac{\ln\left(1+e^{2x}+e^{4x}\right)}{\ln\left(6+e^{8x}+e^{10x}\right)}.$$

   Solution: Observe that $e^{ax}+e^{bx}$ is dominated by the term associated with the larger of a and b. (Meaning that their ratio goes to infinity. This is easily checked.) Thus $\lim_{x\to\infty}(1+e^{2x}+e^{4x})/e^{4x}=1$, and likewise, $(6+e^{8x}+e^{10x})/e^{10x}$ tends to 1. Thus the logs of these ratios tend to zero, or equivalently, the difference between $\ln(6+e^{8x}+e^{10x})$ and $\ln e^{10x}$ tends to zero.

   Our numerator thus amounts to $4x+\text{stuff}$ that goes to zero, and the denominator, to $10x+\text{stuff}$ that goes to zero, so the limit is $2/5$ and that is our answer.

2. Let $f(x)=\tan(\ln(\cos x+\sin x))$ (where defined—there will be points at which the definition of f breaks down).

   (a) Find a formula for $f'(x)$ which holds where f is defined.
   Solution: By the chain rule, the derivative is
$$f'(x)=(\cos x-\sin x)\cdot\frac{1}{\cos x+\sin x}\cdot\sec^2\ln(\cos x+\sin x).$$

   (b) Find the number nearest 0 at which f is not defined. There are two things that can cause f to not be defined. First, the expression whose log is required may be zero or negative. This happens when $\sin x=-\cos x$, or equivalently, when $\tan x=-1$. This occurs at

$-\pi/4$ and at $3\pi/4$ and in general at $-\pi/4 + k\pi$ where k is an integer. The nearest of these is $-\pi/4$ itself.

Second, the log may exist, but its tangent not exist because the log is itself an odd integer multiple of $\pi/2$. Now, we must figure out where $\log(\cos x + \sin x) = \pi/2, -\pi/2, -3\pi/2$ and so forth. The first does not occur because $\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$ by trigonometry, and $\sqrt{2} < \pi/2$. The second does occur, because all we need is that $\cos x + \sin x$ be near enough zero that the log is strongly negative.

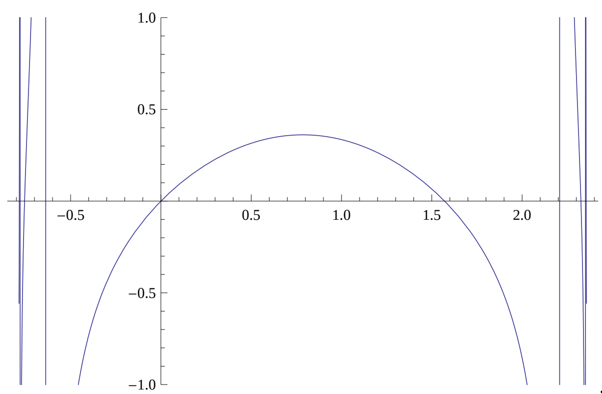
The nearest point will be where $\cos x + \sin x = e^{-\pi/2}$, or equivalently, where $\cos(x - \pi/4) = 2^{-1/2}e^{-\pi/2}$. So, $x - \pi/4 = -\arccos(2^{-1/2}e^{-\pi/2})$. Why the minus sign? Because traditionally, arccos is defined on $[0, \pi]$ and we want a negative number for $x - \pi/4$ because we know our answer is near $-\pi/4$ rather than near $3\pi/4$.

Finally, the answer: $x = \pi/4 - \arccos(2^{-1/2}e^{-\pi/2})$.

(c) Sketch the graph of $f(x)$ on the interval $(-\pi/4, 3\pi/4)$, indicating such discontinuities as may exist. How many are there, in all, on that interval? Why?

Solution: There are infinitely many points of discontinuity, one at each place where $\cos x + \sin x$ hits a small number near enough zero that its log takes the form $-k\pi/2$. These will cram up against the endpoints of our interval $-\pi/4$ and $3\pi/4$.

The graph will thus look like this in the middle, and zooming in, at the edges:



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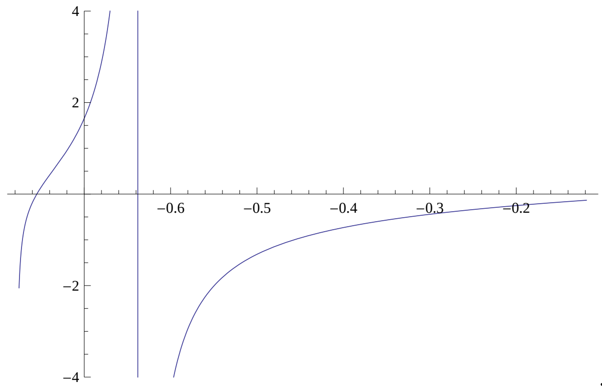


3. Find, accurate to within ± 0.0001, the numerical value as a decimal of the form $a.bcde$ of
$$\int_{x=0}^{1} \frac{1}{x} \sin(x^2)\, dx.$$
Solution: this is a job for Ceres, Goddess of the Harvest! (Here, spelled Series.) The series expansion of $\sin z$ is
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots .$$
(To the question, how am I supposed to know that?—the answer must be, by rote. Like the times table. It's a basic fact that crops up so frequently that knowing it is indispensable.) Anyhow, with some manipulation, $\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots$, and $(1/x)\sin x^2 = x - \frac{x^5}{3!} + \frac{x^9}{5!} - \cdots$. Now this can be integrated term by term, with the result that the required definite integral is given by a rapidly converging, alternating sum with a value $A = \frac{1}{2} - \frac{1}{6\cdot 3!} + \frac{1}{10\cdot 5!} - \cdots$. We have enough terms already and it is time for some arithmetic: $\frac{1}{2} - \frac{1}{36} + \frac{1}{1200} = \frac{1800-100+3}{3600} = \frac{1703}{3600} = 0.4731$, rounded up to the nearest ten-thousandth. The actual value is nearer 0.473042, or if you want lots and lots of places,
$$\begin{aligned} A =& 0.47304153518359150747067665691158982890 6\\ & 1689773690558952357273867833 4\ 351827039896 \end{aligned}$$

4. Let $g(x, y, z) = xy + xyz$.

   (a) Find
   $$\frac{\partial^2 g}{\partial x \partial y}.$$
   Solution: the partial with respect to y is $x + xz$, and the partial of that with respect to x is $1 + z$. The answer is $1 + z$.

(b) Find a point on the unit sphere $x^2+y^2+z^2=1$ at which g is maximal.

Solution: There are a number of ways to go at this. One is that at the best point, the gradient of g is parallel to the gradient of the function whose constant contour gives the constraint surface. Here, that means that $(y+yz, x+xz, xy) = C(x,y,z)$. Where does this (and also, $x^2+y^2+z^2=1$, of course) happen?

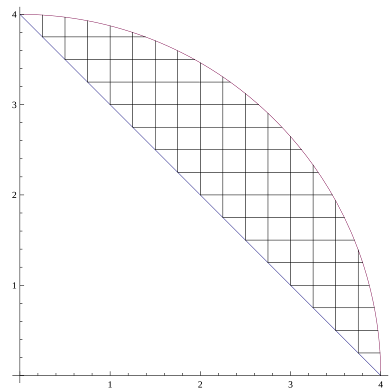
Well, since $(y+yz)/x = C$ and $(x+xz)/y = C$, clearly $y=x$. So now we just need $(x+xz)/x = C = z/x^2$. Equivalently, $(1+z) = z/x^2$, and now recalling that $x^2+x^2+z^2=1$, we have $x^2 = (1-z^2)/2$, so $(1+z) = (1-z^2)/(2z)$. With a little manipulation this becomes $3z^2+2z-1=0$ which has solutions $z=1/3$ and $z=-1$. clearly taking $z=-1$ is a step in the wrong direction (we would do better at $z=0$!), so we take $z=1/3$. That gives $x=\pm 2/3$, which leaves us with two choices: $(2/3, 2/3, 1/3)$ and $(-2/3, -2/3, 1/3)$.

Another approach abstains from using gradients etc. and just proceeds from first principles. Now first principles is good, hearty, solid stuff, but it does tend to involve more work. Sophisticated methods are just slicker. Anyhow, hi ho, it's off to work we go!

Clearly we may as well restrict attention to $x, y, z \geq 0$. Now xy is maximized, subject to a fixed value of x^2+y^2, by taking $y=x$. So we're looking now to maximize $x^2(1+z)$ subject to $2x^2+z^2=1$ and $0 \leq z \leq 1$. That amounts to maximizing $x^2(1+\sqrt{1-2x^2})$ over the interval possible for x, which here is $[0, 1/\sqrt{2}]$. This is a straightforward if somewhat ugly single-variable calculus question. One observes that at $x=0$ the function takes the value 0 while at $x=1$ it takes the value 1, and then the question arises whether there are any points in $(0,1)$ where the derivative is zero. That derivative is $-2x^3/\sqrt{1-2x^2}+2x(1+\sqrt{1-2x^2}$ and factoring out a $1/\sqrt{1-2x^2}$, (which will never be zero), we are solving $2x-6x^3+2x\sqrt{1-2x^2}=0$. Equivalently, (except for the risk of introducing extraneous roots) $4x^2-24x^4+36x^6 = 4x^2-8x^4$.Putting $w=x^2$ and dividing out a factor of 4 this boils down to $-4w^2+9w^3=0$ so $w=4/9$ and $x=2/3$. There is one, it happens, at $x=2/3$. And then we find that $z=1/3$ and we're back to the same answer.

5. Find the centroid of the region L bounded below by the line $y=4-x$

and above by an arc of the circle $x^2 + y^2 = 16$.



The centroid, or center of mass, of a region is got by finding $\overline{x}$ and $\overline{y}$. In this problem, by symmetry, these are equal. So we may as well just find $\overline{y}$. That's the moment M_x of inertia of the figure about the x axis, divided by the area of the figure. And M_x is the double integral of y, taken over the region.

The lower limit of the figure is the line segment $y = 4 - x$, taking x from 0 to 4, while the upper limit is the circle-arc $y = \sqrt{16 - x^2}$. Thus

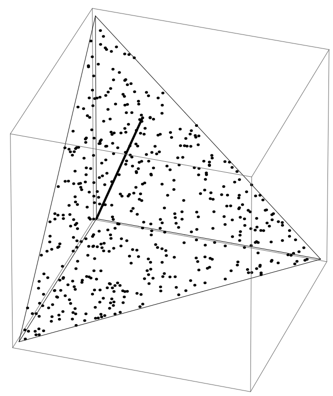
$$M_x = \int_{x=0^4} \int_{y=4-x}^{\sqrt{16-x^2}} y\,dy\,dx.$$

The inner integral evaluates to $\frac{1}{2}((16 - x^2) - (4 - x)^2) = 4x - x^2$, and the outer integral then evaluates to $2x^2 - \frac{1}{3}x^3$ at 4 minus at zero, or $32 - 64/3 = 32/3$. What about the area? We could do another double integral, but it is simpler to use plain old plane geometry. The area of the quarter circle in the positive quadrant is $(1/4)\pi r^2$, and since $r = 4$, that's 4π. The area of the triangle that is not part of L is $(1/2)br$, and here $b = r = 4$ so that's 8. So the area of L itself is $4\pi - 8$, and

$$\overline{y} = \overline{x} = \frac{32/3}{4\pi - 8} = \frac{8}{3\pi - 6}.$$

6. Find the average value of the square of the distance from the origin to points inside the triangle with vertices $(0, 0, 1)$, $(0, 1, 0)$, and $(0, 0, 0)$. (Equivalently, you could think of it like this: imagine a large number of evenly spaced points spread out across that triangle. Imagine that for each, you computed the square of the distance to the origin as shown, and then averaged those numbers. The answer to the original question would

be the limit of this imaginary calculation as ever more points were used.)



We want to integrate the value of $x^2 + y^2 + z^2$ over this triangle, with respect to area, and then divide by the area itself.

It will be simpler to use $dydx$ in place of ds, but we will have to do this for both the integral of the distance squared, and the integral that gives the area. (The ratio of ds over $dydx$ is independent of x and y. Done this way, the 'area' is just 1/2. What about the other integral? We have to get z in terms of x and y. The equation of the plane of our triangle is $x+y+z=1$ (because that equation holds at all three corners) so $z = 1-x-y$. So our integral for distance squared is

$$\int_{x=0}^{1} \int_{y=0}^{1-x} x^2 + y^2 + (1-x-y)^2 \, dy \, dx = \frac{1}{4}.$$

Integrating 1 over the area (1/2) of the region in the x, y plane gives 1/2, and dividing 1/4 by 1/2 we have our answer, 1/2.