TAMU 2018 Freshman-Sophomore Contest

Solutions should include all necessary steps and calculations. This is the Sophomore (2nd year) version; both versions share the same first page.

1. A parabola in the coordinate plane is tangent to the x axis at $(1, 0)$ and to the y axis at $(0, 1)$; its axis is thus the line $y = x$.

   (a) Find the focus of the parabola.

   The focus of a parabola is the point at which all reflections of incoming rays parallel to the axis meet. Here the lines $y = x + 1$ and $y = x-1$ reflect off the parabola at the points $(0, 1)$ and $(1, 0)$ respectively, each at a 45 degree angle from the vertical or horizontal, to run along the line $x + y = 1$. The focus lies on the axis, so $x = y = 1/2$ and the focus is $(1/2, 1/2)$.

   (b) The vertex of the parabola is the point where the axis crosses the parabola. In this case, that point is $(1/4, 1/4)$. Why is that? The directrix of a parabola is a line perpendicular to the axis with the property that the distance from any point on the parabola to the directrix is equal to the distance of that point to the focus. The distance from $(1, 0)$ to $(1/2, 1/2)$ is $1/\sqrt{2}$, so the distance from $(1, 0)$ to the directrix should also be $(1/2, 1/2)$. That puts the directrix going through the point $(1, 0) - (1/2, -1/2) = (1/2, -1/2)$, so the equation of the directrix is $x + y = 0$. The distance from the focus to the vertex is equal to the distance from the vertex to the directrix, which puts the vertex midway between the point at which axis and directrix cross (here, (0,0)), and the focus. Thus the vertex is $(1/4, 1/4)$.

   (c) You may now take as given the previous item. The equation of the parabola can be put in the form $(x + y) = A(x - y)^2 + B$. Find A and B.

   The choice of A and B must work with $(1, 0)$ and $(0, 1)$. This gives $1 = A + B$. Differentiating the given equation gives $1 + y' = 2A(x - y)(1 - y')$. Specializing to $x = 1, y = 0$ gives $1 + 0 = 2A$. Thus $A = B = 1/2$. Another way to finish the calculation is to use the vertex; that gives $(1/4 + 1/4) = B$.

   (d) Find the area of the finite region bounded between the parabola and the line segments joining $(0, 0)$ to $(1, 0)$ and to $(0, 1)$. Solving for y in terms of x and simplifying gives $y = 1 + x - 2\sqrt{x}$, and then $\int_0^1 (1 + x - 2x^{1/2}\, dx = 1 + 1/2 - (4/3) = 1/6$. The area is $1/6$.

2. Prove or disprove that

$$\int_0^\infty \frac{\log(x)\, dx}{1 + x^2} = 0.$$

One way to evaluate definite integrals is to find the antiderivative and evaluate it at the endpoints. This is great if you can find an antiderivative. But that's not always possible. Even if it is possible, your own efforts may not hit upon it. There are other approaches. Here, if you try the substitution $u = 1/x$, then $du = -x^{-2}\,dx$, or equivalently, $dx = -u^{-2}\,du$. This yields

$$A = \int_0^\infty \frac{\log(x)}{1+x^2}\,dx = -\int_\infty^0 \frac{\log(1/u)}{u^2(1+u^{-2})}\,du.$$

When the counting of minus signs from the reversal of order, the explicit minus sign, and the fact that $\log(1/u) = -\log(u)$ is sorted out, the same integral, with only the notational difference that we have u in place of x, appears both as A and as $-A$. Therefore $A = 0$.

3. A one by one square starts out solid green. It is subdivided into four congruent, non-overlapping smaller squares, and the one on the bottom left is colored red.

   The three remaining green squares are each similarly subdivided into nine squares, and in each case, the bottom left sub-square is colored red. At this point, one large square and three smaller squares are red, and there are 24 smaller green squares.

   These green squares are each subdivided into sixteen yet smaller squares, and as before, one of each batch of sixteen is colored red.

   Repeat with 25,36, 49 etc. What portion of the original square remains green throughout?

   The first cut out consumes 1/4 of the original square. The second consumes 1/9 of what is left, so that 8/9 of 3/4 remains. We're now at any area remaining of $\frac{3}{4}\cdot\frac{8}{9} = \frac{4}{6}$. The next batch of red covers 1/16 of what remains, so that the total remaining green is now $\frac{3}{4}\frac{8}{9}\frac{15}{16} = \frac{5}{8}$ and then we have $\frac{5}{8}\cdot 2425 = \frac{6}{10}$. A pattern emerges: after $n-1$ rounds of coloring, the portion that remains green is $(n+1)/2n$. The next round converts that to $\frac{(n+1)(n^2-1)}{2n(n+1)^2} = \frac{n+2}{2n+2}$ and the pattern holds up. The limit is thus 1/2. Half the square remains green and the other half gets colored red.

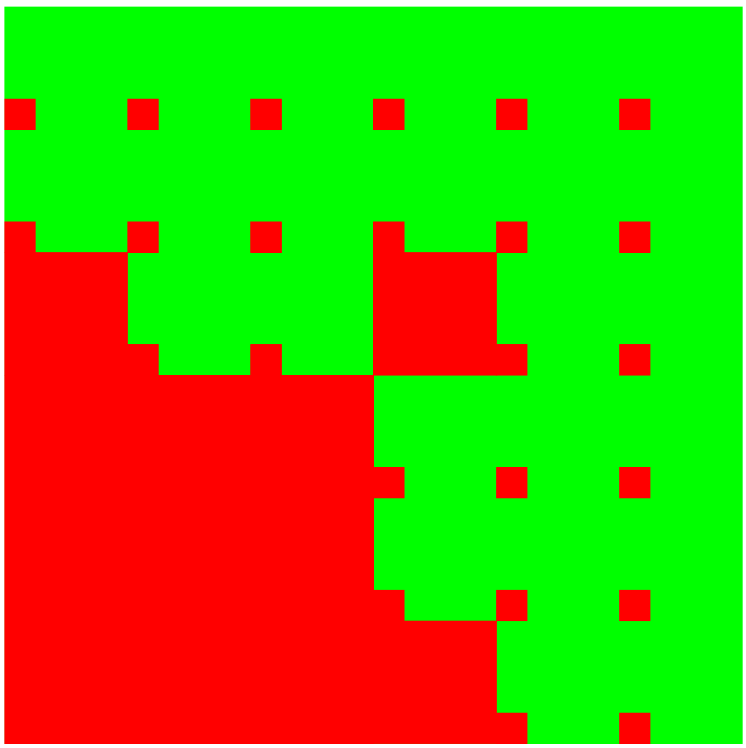

Figure 1: partway through the story

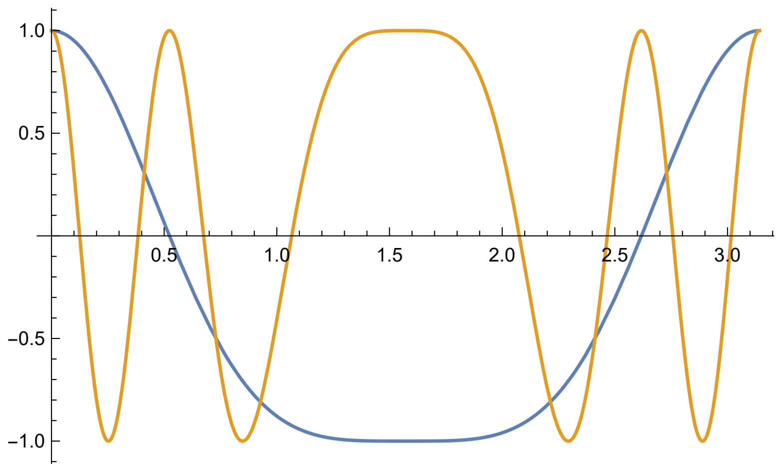

Figure 2: One wave, $\alpha = \pi$,more waves, $\alpha = 4\pi$

4. Let $f_\alpha(x) = \cos(\alpha \sin x)$.

   (a) Sketch the graphs of $f_\pi(x)$ and $f_{4\pi}(x)$ on the interval $0 \le x \le \pi$.

   (b) Let $g(t) = \int_0^\pi f_t(x)\,dx$. You may take as given that differentiation with respect to t may be carried inside the integral. Show that

$$tg''(t) + g'(t) + tg(t) = 0.$$

(Hint: Integration by parts). We begin by by writing out what $tg'' + g' + tg$ is: call it $S(t)$. Then $g'(t) = -\int_0^\pi \sin(t\sin x)\sin x\,dx$ and $g''(t) = -\int_0^\pi \cos(t\sin x)\sin^2 x\,dx$. Thus $S = A + B + C$, say, where

$$A = -\int_0^\pi t\cos(t\sin x)\sin^2 x\,dx,$$

$$B = -\int_0^\pi \sin(t\sin x)\sin x\,dx,$$

and

$$C = \int_0^\pi t\cos(t\sin x)\,dx.$$

In B, which we single out because it involves the off-pattern expression $\sin(t\sin x)$, set $U = \sin(t\sin x)$ and $dV = \sin x\,dx$. Then $dU = t\cos(t\sin x)\cos x\,dx$ and $V = -\cos x$, so

$$\begin{aligned} B &= -\sin(t\sin x)\cos x\big|_{x=0}^{x=\pi} + \int_0^\pi t\cos(t\sin x)\cos^2 x\,dx \\ &= \int_0^\pi t\cos(t\sin x)\cos^2 x\,dx. \end{aligned}$$

Thus

$$S = tg'' + g' + tg = \int_0^\pi t\cos(t\sin x)(-\sin^2 x - \cos^2 x + 1)\,dx = 0.$$

5. Find $\int_0^1 \arctan(\sqrt{x})\,dx$, and express your answer in the form $A\pi + B$, where A and B are rational numbers. First solution: Recall that $\arctan'(u) = 1/(1+u^2)$. From this it follows that $\arctan(u) = \int_0^u ds/(1+s^2)$. Now write the original expression, call it I for 'integral', as

$$I = \int_{x=0}^1 \int_{y=0}^{\sqrt{x}} \frac{1}{1+y^2}\,dy\,dx.$$

Reversing the order of integration gives

$$I = \int_{y=0}^1 \int_{x=y^2}^1 \frac{1}{1+y^2}\,dx\,dy = \int_{y=0}^1 \frac{1-y^2}{1+y^2}\,dy.$$

Now write the numerator as $-1-y^2+2$ and you get $I = -1+2\int_0^1 \frac{1}{1+y^2}\,dy = -1 + 2\arctan 1 = -1 + \pi/2$. So $A = 1/2$ and $B = -1$.

Second solution: make the substitution $x = u^2$, $dx = 2u\,du$. The integral becomes $\int_0^1 2u\tan^{-1} u\,du$. Now integrate by parts, taking $U = \tan^{-1} u$ and $dV = 2u\,du$. You get $u^2 \tan^{-1} u_0^1 - \int_0^1 u^2/(1+u^2)\,du$. The rest of this goes pretty much the same way the other solution did and gives the same answer.

6. Let $x(t)$ and $y(t)$ be twice-differentiable functions of t satisfying the differential equations

$$x'' = \frac{x}{x^2+y^2}, \quad y'' = \frac{y}{x^2+y^2}.$$

Let $\theta(t)$ and $r(t)$ be the polar coordinates of the point with Cartesian coordinates $(x(t), y(t))$. (So, $x(t) = r(t)\cos(\theta(t))$, $y(t) = r(t)\sin(\theta(t))$). Prove that $r^2(t)d\theta/dt$ is constant.

We can rewrite the equations for x'' and y'' to $x'' = \cos(\theta)/r$, $y'' = \sin(\theta)/r$. Now putting everything into $r = r(t)$ and $\theta = \theta(t)$ gives

$$\frac{\cos\theta}{r} = r''\cos\theta - 2r'\theta'\sin\theta - r\cos\theta\theta'^2 = r\sin\theta\theta'',$$
$$\frac{\sin\theta}{r} = r''\sin\theta + 2r'\theta'\cos\theta - r\sin\theta\theta'^2 + r\cos\theta\theta''.$$

Multiply each of these equations by r and multiply the first by $-\sin\theta$ and the second by $\cos\theta$ and add. On the left, this gives 0. On the right, there is a lot of cancellation and with a couple uses of $\cos^2\theta + \sin^2\theta = 1$ the whole thing boils down to $0 = 2rr'\theta' + r^2\theta''$. But this is equivalent to

$$0 = \tfrac{d}{dt}\left(r^2\theta'\right)$$

which shows that $r^2\theta'$ is constant.

Why would anyone expect such a thing to be true? Because it's a lot like conservation of angular momentum. The 'gravity' formula is different, and the result is a different kind of 'angular momentum'.