ASYMPTOTIC DISTRIBUTION OF THE PARTITION CRANK

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ABSTRACT. The partition crank is a statistic on partitions introduced by Freeman Dyson to explain Ramanujan's congruences. In this paper, we prove that the crank is asymptotically equidistributed modulo Q, for any odd number Q. To prove this, we obtain effective bounds on the error term from Rolon's asymptotic estimate for the crank function. We then use those bounds to prove the surjectivity and strict log-subadditivity of the crank function.

1. INTRODUCTION AND STATEMENT OF RESULTS

A partition λ of $n \in \mathbb{N}$ is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, such that $\lambda_1 + \ldots + \lambda_k = n$. Each λ_i is called a *part* of the partition λ and S_n is the set of all partitions of n. The partition function, $p(n) := \#S_n$, counts the number of distinct partitions of n.

In 1918, Hardy and Ramanujan [4] gave the following asymptotic formula for $p(n)$:

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Using $p(n)$, Ramanujan's congruences state that for any $l \in \mathbb{N}$, we have:

$$p(5l+4) \equiv 0 \pmod 5$$

$$p(7l+5) \equiv 0 \pmod 7$$

$$p(11l+6) \equiv 0 \pmod{11},$$

as well as several other congruences modulo any number of the form $5^a 7^b 11^c$, where $a, b, c \in \mathbb{N}$.

Freeman Dyson [1] conjectured that the congruences modulo 5 and 7 could be proved by a function he called the *rank*. The rank of a partition λ is defined to be

$$\text{rank}(\lambda) := \lambda_1 - k.$$

Let $N(r, Q; n)$ be the number of partitions of n with rank congruent to r modulo Q. Dyson conjectured that:

(i) for each $r \pmod 5$, $N(r, Q; 5l+4) = \frac{1}{5} p(5l+4)$.
(ii) for each $r \pmod 7$, $N(r, Q; 7l+5) = \frac{1}{7} p(7l+5)$.

In 1954, Atkin and Swinnerton-Dyer [2] proved this conjecture. Unfortunately, the rank fails to show the congruence modulo 11. Dyson conjectured the existence of another statistic which he called the *crank* which would explain all three congruences. In 1988, Andrews and Garvan [1] found such a crank.

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Let $o(\lambda)$ be the number of 1's in λ and $\nu(\lambda)$ be the number of parts of λ larger than $o(\lambda)$. The crank of λ is then defined to be

$$\mathrm{crank}(\lambda) := \begin{cases} \lambda_1 & \text{if } o(\lambda) = 0 \\ \nu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

Let $M(r,Q;n)$ be the number of partitions of n with crank r modulo Q. Ramanujan's congruences are explained by the strict equidistribution of $M(r,Q;n)$ over all $r \pmod{Q}$ for certain Q and n. Here we show that $M(r,Q;n)$ is asymptotically equidistributed over r for all odd Q as $n \to \infty$.

In [8], Rolon gave an asymptotic expression for $M(r,Q;n)$ with an error term which is $O(n^{\varepsilon})$. Here we refine his analysis to get an effective bound on the error with explicit constants. We use this bound to prove asymptotic equidistribution.

Let $\mu(n) := \sqrt{24n-1}$. Then we state one of main results.

Theorem 1. *Let $0 \le r < Q$ with Q an odd integer. Then we have*

$$\frac{M(r,Q;n)}{p(n)} = \frac{1}{Q} + R(r,Q;n),$$

where when $Q < 11$ we have

$$|R(r,Q;n)| \le 10^5(40.93Q + 6.292)e^{-\left(1-\frac{1}{Q}\right)\frac{\pi\mu(n)}{6}} n^{\frac{11}{8}}$$

and when $Q \ge 11$ we have

$$|R(r,Q;n)| \le 10^5(40.93Q + 6.292)e^{-\left(1-\sqrt{1+12\left(\frac{1}{Q^2}-\frac{1}{Q}\right)}\right)\frac{\pi\mu(n)}{6}} n^{\frac{11}{8}}.$$

It follows immediately that the cranks are asymptotically equidistributed modulo Q.

Corollary 1. *Let $0 \le r < Q$ with Q an odd integer. Then we have*

$$\frac{M(r,Q;n)}{p(n)} \longrightarrow \frac{1}{Q}$$

as $n \to \infty$.

Corollary 1 can be seen as an analogue to Dirichlet's theorem about the equidistribution of primes over all residues r for any modulus Q with $(r,Q) = 1$. Because of this we will also use Theorem 1 to find an analogue to Linnik's theorem which gives an upper bound for the smallest prime in each residue class.

Theorem 2. *Let Q be an odd integer and when $Q \ge 11$ we define the constant*

$$C_Q := \frac{(1.93 \times 10^{59})(40.93Q^2 + 6.292Q)^8}{\left(\pi - \pi\sqrt{1+12\left(\frac{1}{Q^2}-\frac{1}{Q}\right)}\right)^{24}} + 1. \tag{1}$$

Then we have

$$M(r,Q;n) > 0,$$

if $Q < 11$ and $n \ge 263$, or if $Q \ge 11$ and $n \ge C_Q$.

The restriction to Q odd in Theorem 2 can be removed by a different, combinatorial argument. We state the following theorem.

Theorem 3. *If $Q \geq 11$ is odd and $n \geq \frac{Q+1}{2}$ or $Q \geq 8$ is even and $n \geq \frac{Q}{2}+2$, then we have*

$$M(r,Q;n) > 0.$$

Also, Bessenrodt and Ono [3] prove strict log-subadditivity of the partition function, and later Locus Dawsey and Masri [6] prove a similar result for the spt-function. Here we show an analogue for the crank counting function by using Theorem 1 to produce effective constants for a and b.

Theorem 4. *Given any residue $r \pmod{Q}$ where Q is odd, we have*

$$M(r,Q;a+b) < M(r,Q;a)M(r,Q;b),$$

if $Q < 11$ and $a,b \geq 396$ or if $Q \geq 11$ and $a,b \geq C_Q$, where C_Q is defined in (1).

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2. Effective Asymptotic Formula for $M(r,Q;n)$

In [8, 9], Rolon gives an asymptotic formula for $M(r,Q;n)$. Here we refine his analysis and give an asymptotic formula with an effective bound on the error term. We begin by stating a few necessary definitions.

Let

$$\omega_{h,k} := \exp(\pi i s(h,k)),$$

where the Dedekind sums $s(h,k)$ are defined by

$$s(h,k) := \sum_{u \pmod{k}} \left(\left(\frac{u}{k}\right)\right)\left(\left(\frac{hu}{k}\right)\right).$$

Here $((\cdot))$ is the sawtooth function defined by

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Let $0 \leq h < k$ be relatively prime integers. Let $0 < r < Q$ be relatively prime integers where Q is odd. Let h' be a solution to the congruence $hh' \equiv -1 \pmod{k}$ if k is odd and $hh' \equiv -1 \pmod{2k}$ if k is even. Let $c_1 := \frac{c}{(c,k)}$ and $k_1 := \frac{k}{(c,k)}$. Let l be the minimal positive solution to $l \equiv ak_1 \pmod{c_1}$. For $m,n \in \mathbb{Z}$ we define:

$$\widetilde{B}_{a,c,k}(n,m) := (-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h \pmod{k} \\ (h,k)=1}} \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} e^{\frac{-\pi i a^2 k_1 h'}{c}} e^{\frac{2\pi i}{k}(nh+mh')},$$

where the sum runs over all primitive residue classes modulo k.

For the case $c \nmid k$ we define:

$$D_{a,c,k}(m,n) := (-1)^{ak+l} \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h \pmod{k} \\ (h,k)=1}} \omega_{h,k} e^{\frac{2\pi i}{k}(nh+mh')},$$

where l is the solution to $l \equiv ak_1 \pmod{c_1}$.

In order to provide certain bounds, Rolon defines the following:

$$\delta^{i}_{a,c,k,r} := \begin{cases} -(\frac{1}{2}+r)\frac{l}{c_1}+\frac{1}{2}(\frac{l}{c_1})^2+\frac{1}{24} & \text{if } i=+, \\ \frac{l}{2c_1}+\frac{1}{2}(\frac{l}{c_1})^2-\frac{23}{24}-r(1-\frac{l}{c_1}) & \text{if } i=-, \end{cases}$$

$$\delta_0 := \frac{1}{2Q^2}-\frac{1}{2Q}+\frac{1}{24}<\frac{1}{24},$$

and

$$m^{+}_{a,c,k,r} := \frac{1}{2c_1^2}(-a^2k_1^2+2lak_1-ak_1c_1-l^2+lc_1-2ark_1c_1+2lc_1r),$$

$$m^{-}_{a,c,k,r} := \frac{1}{2c_1^2}(-a^2k_1^2+2lak_1-ak_1c_1-l^2+2c_1^2r-2lrc_1+2ark_1c_1+2lc_1+2c_1^2-ak_1c_1).$$

Note that $\delta^{\pm}_{a,Q,k,r} \leq \delta_0 < \frac{1}{24}$.

Rolon obtains an asymptotic formula for $M(r,Q;n)$ using the circle method. First, Rolon defines the generating function

$$C(w,q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)w^m q^n \tag{2}$$

where $M(m,n)$ is the number of partitions of n with crank m. In order to use the modular properties of this function, Rolon plugs in a root of unity for w and studies the coefficients of q. He defines

$$C(e^{2\pi i\frac{j}{k}},q) =: \sum_{n=0}^{\infty}\tilde{A}\left(\frac{j}{k},n\right)q^n. \tag{3}$$

Rolon then uses the circle method to find an asymptotic formula for $\tilde{A}\left(\frac{j}{k},n\right)$, and uses the identity

$$M(r,Q;n) = \frac{1}{Q}\sum_{j=0}^{Q-1}\zeta^{-jr}\tilde{A}\left(\frac{j}{Q},n\right) \tag{4}$$

to get an asymptotic formula for $M(r,Q;n)$. Note that $\tilde{A}(\frac{0}{Q},n)=p(n)$.

Rolon [8, 9] gives the following asymptotic formula for $\tilde{A}\left(\frac{j}{Q},n\right)$:

$$\tilde{A}\left(\frac{j}{Q},n\right) = \frac{4\sqrt{3i}}{\mu(n)}\sum_{\substack{Q|k \\ k\leq\sqrt{n}}}\frac{\tilde{B}_{j,Q,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\mu(n)}{6k}\right) \tag{5}$$

$$+\frac{8\sqrt{3}\sin(\frac{\pi j}{Q})}{\mu(n)}\sum_{\substack{k,s \\ Q\nmid k \\ \delta^{i}_{j,Q,k,s}>0 \\ i\in\{+,-\}}}\frac{D_{j,Q,k}(-n,m^{i}_{j,Q,k,s})}{\sqrt{k}}\sinh\left(\sqrt{24\delta^{i}_{j,Q,k,s}}\frac{\pi\mu(n)}{6k}\right)+O(n^{\varepsilon}),$$

which when plugged into equation (4) gives

$$M(r,Q;n) = \frac{1}{Q}p(n) + \frac{1}{Q}\sum_{j=1}^{Q-1}\zeta_Q^{-rj}\frac{4\sqrt{3i}}{\mu(n)}\sum_{\substack{Q|k\\ k\le\sqrt{n}}}\frac{\tilde{B}_{j,Q,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\mu(n)}{6k}\right) \tag{6}$$

$$+\frac{1}{Q}\sum_{j=1}^{Q-1}\zeta_Q^{-rj}\frac{8\sqrt{3}\sin(\frac{\pi j}{Q})}{\mu(n)}\sum_{\substack{k,s\\ Q\nmid k\\ \delta^i_{j,Q,k,s}>0\\ i\in\{+,-\}}}\frac{D_{j,Q,k}(-n,m^i_{j,Q,k,s})}{\sqrt{k}}\sinh\left(\sqrt{24\delta^i_{j,Q,k,s}}\frac{\pi\mu(n)}{6k}\right) + O(n^{\varepsilon}).$$

Proof of Theorem 1. Rolon breaks the $O(n^{\varepsilon})$ error term from the calculation of $\tilde{A}(\frac{j}{Q},n)$ into six pieces: $S_{err}, S_{1err}, S_{2err}, T_{err}$, and the contributions of error from certain integrals which we will call $\Sigma_1 I_{err}$ and $\Sigma_2 I_{err}$. He provides bounds on each of those pieces, which we can then refine and sum up to get bounds on the error in the formula for $\tilde{A}(\frac{j}{Q},n)$. Then using equation (4) and the triangle inequality, we can get our desired bound on $|R(r,Q;n)|$.

Fix odd integers j and Q. We will bound the error coming from $\tilde{A}(\frac{j}{Q},n)$. Rolon provides the following bounds:

$$|S_{err}| \le \frac{2e^{2\pi+\frac{\pi}{24}}|\sin(\frac{\pi j}{Q})|(c_2+2(1+|\cos(\frac{\pi}{Q})|)c_1(1+c_2))n^{\frac{1}{4}}\left(1+\log\left(\frac{Q-1}{2}\right)\right)}{\pi(1-\frac{\pi^2}{24})Q},$$

$$|T_{err}| \le 16e^{2\pi}f(Q)n^{\frac{1}{4}}\left|\sin\left(\frac{\pi j}{Q}\right)\right|,$$

where

$$f(Q) := \frac{1+c_2e^{\pi\delta_0}}{1-e^{\frac{-\pi}{Q}}} + e^{\pi\delta_0}c_1(1+c_2) + \frac{e^{\pi\delta_0}(c_2+1)c_3}{2},$$

and where the c_i are constants defined in Rolon's paper. We have the approximations $c_1 \le 0.046$, $c_2 \le 1.048$, and $c_3 \le 0.001$. Also,

$$|S_{1err}| \le \frac{8e^{2\pi+\frac{\pi}{12}}(1+\log(\frac{Q-1}{2}))n^{\frac{1}{4}}}{\pi(1-\frac{\pi^2}{24})Q},$$

$$|S_{2err}| \le 32e^{2\pi}n^{\frac{1}{4}}\left|\sin\left(\frac{\pi j}{Q}\right)\right|\frac{e^{2\pi\delta_0}}{1-e^{\frac{-2\pi}{Q}}},$$

$$|\Sigma_1 I_{err}| \le \frac{4\left(\frac{4}{3}+2^{\frac{5}{4}}\right)\left|\sin\left(\frac{\pi j}{Q}\right)\right|\left(1+\log\left(\frac{Q-1}{2}\right)\right)e^{2\pi+\frac{\pi}{12}}n^{\frac{3}{8}}}{\pi\left(1-\frac{\pi^2}{24}\right)Q},$$

$$|\Sigma_2 I_{err}| \le 8\left(\frac{4}{3}+2^{\frac{5}{4}}\right)\left|\sin\left(\frac{\pi j}{Q}\right)\right|\frac{e^{2\pi\delta_0+2\pi}}{1-e^{-\frac{2\pi}{Q}}}.$$

Now we estimate some of the expressions in those bounds in order to simplify them:

(i) $\left|\sin\left(\frac{\pi j}{Q}\right)\right| \leq 1$,
(ii) $\frac{(1+\log(\frac{Q-1}{2}))}{\pi(1-\frac{\pi^2}{24})Q} \leq 0.1902$,
(iii) $\left(\frac{4}{3}+2^{\frac{5}{4}}\right) \leq 3.712$,
(iv) $\frac{1}{1-e^{-\frac{\pi}{Q}}} \leq \pi Q$,
(v) $\frac{1}{1-e^{-\frac{2\pi}{Q}}} \leq 2\pi Q$.

In order to prove these last two bounds, let $g(Q) := \frac{1}{1-e^{-\frac{b}{Q}}}$. This satisfies the differential inequality $g'(Q) < b\frac{g(Q)^2}{Q^2}$. The function $h(Q) = bQ$ satisfies $h'(Q) = b\frac{h(Q)^2}{Q^2}$, and in the case of $b = \pi$ and $b = 2\pi$ we have $h(1) > g(1)$ and $h'(1) > g'(1)$. This implies that $h(Q) > g(Q)$ for all $Q \geq 1$, as desired.

Now we simplify the bounds given by Rolon:

$$|S_{err}| \leq 330.9n^{\frac{1}{4}},$$

$$|T_{err}| \leq (59071Q+930.05)n^{\frac{1}{4}},$$

$$|S_{1err}| \leq 1059n^{\frac{1}{4}},$$

$$|S_{2err}| \leq 22306n^{\frac{1}{4}},$$

$$|\Sigma_1 I_{err}| \leq 1965n^{\frac{3}{8}},$$

$$|\Sigma_2 I_{err}| \leq 113883Q.$$

Summing these all up gives the total contribution of the $O(n^{\varepsilon})$ error term to $\tilde{A}(\frac{j}{Q},n)$. We then use equation (4) to get the contribution of the error term to $M(r,Q;n)$. However, after applying the triangle inequality these two bounds will be the same except for a factor of $\frac{Q-1}{Q}$, which we will round up to 1 for simplicity. So, the bound for the $O(n^{\varepsilon})$ error term of $M(r,Q;n)$ is

$$(172954Q+26591)n^{\frac{3}{8}}.$$

Now we will bound the main terms from the formula for $M(r,Q;n)$. We use the following bounds from [8]:

(i) $\widetilde{B}_{j,Q,k}(-n,0) \leq \frac{2k\left(1+\log\frac{Q-1}{2}\right)}{\pi\left(1-\frac{\pi^2}{24}\right)} \leq 0.3804kQ$,
(ii) $D_{j,Q,k}(-n,m^i_{j,Q,k,s}) \leq k$,
(iii) $\sum_{\substack{Q|k \\ k\leq\sqrt{n}}} k^{\frac{1}{2}} \leq \frac{2}{3Q}n^{\frac{3}{4}}$,
(iv) $\sinh\left(\sqrt{24\delta^j_{j,Q,k,s}\frac{\pi\mu(n)}{6k}}\right) \leq \frac{1}{2}e^{\sqrt{24\delta_0}\frac{\pi\mu(n)}{6k}}$.

Additionally, Rolon states on [8, page 35] that for fixed k the number of terms in the sum

$$\sum_{\substack{k,s\\ Q\nmid k\\ \delta^i_{j,Q,k,s}>0\\ i\in\{+,-\}}}$$

is bounded by $\frac{Q+18}{24}$. An integral comparison tells us

$$\sum_{\substack{Q\nmid k\\ k\leq\sqrt{n}}} k^{\frac{1}{2}} \leq \frac{2}{3}n^{\frac{3}{4}}.$$

Now by the triangle inequality,

$$\begin{aligned}
&\left|\frac{1}{Q}\sum_{j=1}^{Q-1}\zeta_Q^{-rj}\frac{4\sqrt{3}i}{\mu(n)}\sum_{\substack{Q|k\\ k\leq\sqrt{n}}}\frac{\tilde{B}_{j,Q,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\mu(n)}{6k}\right)\right|\\
&\leq \frac{4\sqrt{3}}{\mu(n)}\sinh\left(\frac{\pi\mu(n)}{6Q}\right)\sum_{\substack{Q|k\\ k\leq\sqrt{n}}}0.3804\sqrt{k}Q\\
&\leq 1.757\frac{1}{\mu(n)}\frac{1}{2}e^{\frac{\pi\mu(n)}{6Q}}n^{\frac{3}{4}}\\
&\leq 0.8785e^{\frac{\pi\mu(n)}{6Q}}n^{\frac{1}{4}}.
\end{aligned}$$

Now we bound the other main term:

$$\begin{aligned}
&\left|\frac{1}{Q}\sum_{j=1}^{Q-1}\zeta_Q^{-rj}\frac{8\sqrt{3}\sin(\frac{\pi j}{Q})}{\mu(n)}\sum_{\substack{k,s\\ Q\nmid k\\ \delta^i_{j,Q,k,s}>0\\ i\in\{+,-\}}}\frac{D_{j,Q,k}(-n,m^i_{j,Q,k,s})}{\sqrt{k}}\sinh\left(\sqrt{24\delta^i_{j,Q,k,s}}\frac{\pi\mu(n)}{6k}\right)\right|\\
&\leq \frac{8\sqrt{3}}{\mu(n)}\sinh\left(\sqrt{24\delta_0}\frac{\pi\mu(n)}{6}\right)\sum_{\substack{k,s\\ Q\nmid k\\ \delta^i_{j,Q,k,s}>0\\ i\in\{+,-\}}}\sqrt{k}\\
&\leq \frac{8\sqrt{3}}{\mu(n)}\frac{1}{2}e^{\sqrt{24\delta_0}\frac{\pi\mu(n)}{6}}\frac{Q+18}{24}\cdot\frac{2}{3}n^{\frac{3}{4}}\\
&\leq (0.1924Q+3.464)e^{\sqrt{24\delta_0}\frac{\pi\mu(n)}{6}}n^{\frac{1}{4}}.
\end{aligned}$$

From [5] we get the following lower bound for $p(n)$:

$$p(n) > \frac{\sqrt{3}}{12n}\left(1 - \frac{1}{\sqrt{n}}\right)e^{\frac{\pi\mu(n)}{6}}.$$

We also note that for $n \geq 2$

$$\frac{1}{1-\frac{1}{\sqrt{n}}} \leq \frac{1}{1-\frac{1}{\sqrt{2}}} \leq 3.415.$$

Now finally by the triangle inequality,

$$\begin{aligned}
|R(r,Q;n)| &= \left|\frac{M(r,Q;n)}{p(n)} - \frac{1}{Q}\right| \\
&\leq \left|\frac{1}{p(n)}0.8785e^{\frac{\pi\mu(n)}{6Q}}n^{\frac{1}{4}} + \frac{1}{p(n)}(0.1924Q+3.464)e^{\sqrt{24\delta_0}\frac{\pi\mu(n)}{6}}n^{\frac{1}{4}} + \frac{1}{p(n)}(172954Q+26591)n^{\frac{3}{8}}\right| \\
&\leq 20.79e^{\left(\frac{1}{Q}-1\right)\frac{\pi\mu(n)}{6}}n^{\frac{5}{4}} + (4.553Q+81.96)e^{\left(\sqrt{24\delta_0}-1\right)\frac{\pi\mu(n)}{6}}n^{\frac{5}{4}} + 10^5(40.93Q+6.292)e^{-\frac{\pi\mu(n)}{6}}n^{\frac{11}{8}}.
\end{aligned}$$

This is a sum of three terms each with similar factors. In order to combine this into an upper bound which can be worked with we take the sum of all three coefficients, the highest order exponential, and the highest power of n from the three terms and put them together in one term. This gives the bounds in the statement of the theorem. We have to break up the $Q < 11$ and $Q \geq 11$ cases because that is the point at which $\frac{1}{Q} - 1$ is overtaken by $\sqrt{24\delta_0} - 1$. Note that the third term has far larger coefficients but also a much faster decaying exponential term, so a lot of accuracy is lost when combining this term with the others. □

3. SURJECTIVITY

We can think of the crank as a function that maps the set of partitions of n, S_n, to the set of integers, $\mathbb{Z}$. We can then take the reduction of this map modulo Q to get a function from S_n to $\mathbb{Z}/Q\mathbb{Z}$. It is natural to ask for which n this map is surjective. This is an analogue of Linnik's theorem for the least prime in an arithmetic progression.

Proof of Theorem 2. In order to prove this map is surjective, it is sufficient to prove that

$$|R(r,Q;n)| < \frac{1}{Q},$$

because this implies $M(r,Q;n) > 0$.

By our bounds on $|R(r,Q;n)|$, when $Q < 11$ we need

$$10^5(40.93Q+6.292)e^{-\left(1-\frac{1}{Q}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}} < \frac{1}{Q},$$

and when $Q \geq 11$ we need

$$10^5(40.93Q+6.292)e^{-\left(1-\sqrt{1+12\left(\frac{1}{Q^2}-\frac{1}{Q}\right)}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}} < \frac{1}{Q}.$$

First assume that $Q < 11$. Then in order to show the inequality

$$10^5(40.93Q+6.292)e^{-\left(1-\frac{1}{Q}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}} < \frac{1}{Q},$$

it suffices to show that

$$10^5(40.93\times 11+6.292)e^{-\left(1-\frac{1}{3}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}}<\frac{1}{11}. \tag{7}$$

By a short computation, we find that (7) holds when $n\geq 263$.

Hence, it follows that if $Q<11$ and $n\geq 263$, then

$$|R(r,Q;n)|<\frac{1}{Q}.$$

Next, we deal with the case $Q\geq 11$. It suffices to show that

$$|R(r,Q;n)|\leq 10^5(40.93Q+6.292)e^{-\left(1-\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}}<\frac{1}{2Q},$$

where we replaced $1/Q$ with $1/2Q$ since we will need this inequality in Section 4. To verify the inequality, it is equivalent to show that

$$\frac{e^{\left(1-\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)\frac{\pi}{6}\mu(n)}}{n^{\frac{11}{8}}}>2\times 10^5Q(40.93Q+6.292). \tag{8}$$

Moreover, we recall the following inequality [7, Eq. 4.5.13]

$$e^x>\left(1+\frac{x}{y}\right)^y,\qquad x,y>0. \tag{9}$$

Hence, by taking $y=3$ in (9), we get

$$\begin{aligned}\frac{e^{\left(1-\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)\frac{\pi}{6}\mu(n)}}{n^{\frac{11}{8}}}&>\frac{1}{n^{\frac{11}{8}}}\left(1+\left(1-\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)\frac{\pi}{18}\mu(n)\right)^3\\&>\frac{\pi^3(24n-1)^{\frac{3}{2}}}{18^3n^{\frac{11}{8}}}\left(1-\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)^3.\end{aligned}$$

By combining (8), it suffices to show that

$$\frac{(24n-1)^{\frac{3}{2}}}{n^{\frac{11}{8}}}>\frac{2\times 10^5\times 18^3Q(40.93Q+6.292)}{\left(\pi-\pi\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)^3}. \tag{10}$$

Also, if $n\geq 2$, then we have

$$\frac{(24n-1)^{\frac{3}{2}}}{n^{\frac{11}{8}}}>\frac{24^{\frac{3}{2}}(n-1)^{\frac{3}{2}}}{n^{\frac{11}{8}}}=24^{\frac{3}{2}}\left(1-\frac{1}{n}\right)^{\frac{11}{8}}(n-1)^{\frac{1}{8}}\geq\frac{24^{\frac{3}{2}}}{2^{\frac{11}{8}}}(n-1)^{\frac{1}{8}}.$$

Hence, by a simple calculation, if we choose the constant

$$C_Q:=\frac{(1.93\times 10^{59})(40.93Q^2+6.292Q)^8}{\left(\pi-\pi\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)^{24}}+1,$$

then (10) holds when $n\geq C_Q>2$. This completes the proof. ☐

Remark. From our estimation, the exponent y in (9) controls the magnitude of C_Q. Hence, it is not hard to see that the constant C_Q will be $C_Q \asymp Q$, as y is sufficiently large.

There is a different, combinatorial method that works for all Q.

Proof of Theorem 3. It is well known that over all partitions of $n \geq 6$, the crank takes on exactly the values $-n$ through n except for $-n+1$ and $n-1$. For even Q and $n \geq \frac{Q}{2}+2$ this means the crank takes on at least Q consecutive values, so the crank maps onto each residue class. For odd Q and $n = \frac{Q+1}{2}$, the residues $\frac{Q\pm1}{2}$ are mapped onto by $-n$ and n, and all the other residues are mapped onto by $-n+2$ through $n-2$. For $n > \frac{Q+1}{2}$, the crank takes on at least Q consecutive values. $\square$

4. Strict log-Subadditivity for Crank Functions

Bessenrodt and Ono [3] showed that if $a, b \geq 1$ and $a+b \geq 9$, then

$$p(a+b) < p(a)p(b).$$

Also, Locus Dawsey and Masri [6] showed the following similar result for the spt-function,

$$\mathrm{spt}(a+b) < \mathrm{spt}(a)\mathrm{spt}(b),$$

for $(a,b) \neq (2,2)$ or $(3,3)$.

Now, we prove Theorem 4 which is analogous result for the crank counting function.

Proof of Theorem 4. We first deal with the case $Q < 11$. By our bounds on $|R(r,Q;n)|$, when $Q < 11$ we have

$$L(Q,n) < M(r,Q;n) < U(Q,n),$$

where

$$L(Q,n) := p(n)\left(\frac{1}{Q} - 10^5(40.93Q+6.292)e^{-\left(1-\frac{1}{Q}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}}\right),$$
$$U(Q,n) := p(n)\left(\frac{1}{Q} + 10^5(40.93Q+6.292)e^{-\left(1-\frac{1}{Q}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}}\right).$$

Moreover, by $3 \leq Q < 11$ and $n \geq 263$, we have

$$L(Q,n) > p(n)\left(\frac{1}{11} - 10^5(40.93\times 11+6.292)e^{-\frac{\pi}{9}\mu(n)}n^{\frac{11}{8}}\right) > (0.00306)p(n).$$

Similarly, we get

$$U(Q,n) < p(n)\left(\frac{1}{3} + 10^5(40.93\times 11+6.292)e^{-\frac{\pi}{9}\mu(n)}n^{\frac{11}{8}}\right) < (1.10213\times 10^7)p(n).$$

Hence, if $n \geq 263$, then we have

$$(0.00306)p(n) < M(r,Q;n) < (1.10213\times 10^7)p(n).$$

On the other hand, Lehmer [5] gives the following bounds for $p(n)$:

$$\frac{\sqrt{3}}{12n}\left(1-\frac{1}{\sqrt{n}}\right)e^{\frac{\pi}{6}\mu(n)} < p(n) < \frac{\sqrt{3}}{12n}\left(1+\frac{1}{\sqrt{n}}\right)e^{\frac{\pi}{6}\mu(n)}.$$

Together these give the bounds.

$$(0.00306)\frac{\sqrt{3}}{12n}\left(1-\frac{1}{\sqrt{n}}\right)e^{\frac{\pi}{6}\mu(n)} < M(r,Q;n) < (1.10213\times 10^7)\frac{\sqrt{3}}{12n}\left(1+\frac{1}{\sqrt{n}}\right)e^{\frac{\pi}{6}\mu(n)}. \tag{11}$$

Now, we follow the argument in [6, Section 6] and let $b = Ca$ for some $C \geq 1$. Then by (11), it follows that

$$M(r,Q,a)M(r,Q,b) > (0.00306)^2\frac{1}{48Ca^2}\left(1-\frac{1}{\sqrt{a}}\right)\left(1-\frac{1}{\sqrt{Ca}}\right)e^{\frac{\pi}{6}(\mu(a)+\mu(Ca))},$$

and

$$M(r,Q,a+b) < (1.10213\times 10^7)\frac{\sqrt{3}}{12(a+Ca)}\left(1+\frac{1}{\sqrt{a+Ca}}\right)e^{\frac{\pi}{6}\mu(a+Ca)}.$$

It suffices to show that

$$T_a(C) > \log\left(V_a(C)\right)+\log\left(S_a(C)\right),$$

where

$$T_a(C) := \frac{\pi}{6}(\mu(a)+\mu(Ca)-\mu(a+Ca)),$$
$$S_a(C) := \frac{1+\frac{1}{\sqrt{a+Ca}}}{\left(1-\frac{1}{\sqrt{a}}\right)\left(1-\frac{1}{\sqrt{Ca}}\right)},$$
$$V_a(C) := \frac{(1.10213\times 10^7)}{(0.00306)^2}\frac{4\sqrt{3}Ca}{C+1}.$$

As functions of C, it can be shown that $T_a(C)$ is increasing and $S_a(C)$ is decreasing for $C \geq 1$, and by combining

$$V_a(C) < \frac{(1.10213\times 10^7)}{(0.00306)^2}4\sqrt{3}a,$$

it suffices to show that

$$\begin{aligned} T_a(1) = \frac{\pi}{6}(2\mu(a)-\mu(2a)) &> \log\left(\frac{(1.10213\times 10^7)}{(0.00306)^2}4\sqrt{3}a\right)+\log\left(\frac{1+\frac{1}{\sqrt{2a}}}{\left(1-\frac{1}{\sqrt{a}}\right)^2}\right) \\ &= \log\left(\frac{(1.10213\times 10^7)}{(0.00306)^2}4\sqrt{3}a\right)+\log\left(S_a(1)\right). \end{aligned} \tag{12}$$

By computing the values $T_a(1)$, $S_a(1)$ and $V_a(C)$, we find that (12) holds for all $a \geq 396$.

Hence, if $Q < 11$ and $a, b \geq 396$, then we have

$$M(r,Q;a+b) < M(r,Q;a)M(r,Q;b).$$

Next, we deal with the case $Q \geq 11$. By our bounds on $|R(r,Q;n)|$, when $Q \geq 11$ we have

$$L_2(Q,n) < M(r,Q;n) < U_2(Q,n),$$

where

$$L_2(Q,n) := p(n)\left(\frac{1}{Q} - 10^5(40.93Q+6.292)e^{-\left(1-\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}}\right),$$
$$U_2(Q,n) := p(n)\left(\frac{1}{Q} + 10^5(40.93Q+6.292)e^{-\left(1-\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}}\right).$$

By the proof of Theorem 2, we know that if $n \geq C_Q$, then we have

$$|R(r,Q;n)| \leq 10^5(40.93Q+6.292)e^{-\left(1-\sqrt{1+12(\frac{1}{Q^2}-\frac{1}{Q})}\right)\frac{\pi}{6}\mu(n)}n^{\frac{11}{8}} < \frac{1}{2Q}.$$

It follows that

$$\left(\frac{1}{2Q}\right)p(n) < M(r,Q;n) < \left(\frac{3}{2Q}\right)p(n).$$

By the same argument of the case $Q < 11$, we need to show that for any $b = Ca$ for some $C \geq 1$,

$$T_a(C) > \log\left(W_a(C)\right) + \log\left(S_a(C)\right),$$

where

$$W_a(C) := (6Q)\frac{4\sqrt{3}Ca}{C+1}.$$

Moreover, by the trivial bound

$$W_a(C) < 24\sqrt{3}Qa,$$

and the same argument, it suffices to show that

$$T_a(1) = \frac{\pi}{6}(2\mu(a) - \mu(2a)) > \log(24\sqrt{3}Qa) + \log\left(\frac{1+\frac{1}{\sqrt{2a}}}{\left(1-\frac{1}{\sqrt{a}}\right)^2}\right) = \log(24\sqrt{3}Qa) + \log\left(S_a(1)\right). \tag{13}$$

On the other hand, if $a \geq 2$, then we get

$$\log(24\sqrt{3}Qa) + \log\left(S_a(1)\right) < \log(24\sqrt{3}Qa) + \log\left(\frac{1+\frac{1}{\sqrt{4}}}{\left(1-\frac{1}{\sqrt{2}}\right)^2}\right) < \log(432Qa). \tag{14}$$

Also, if $a \geq (432Q)^2 \geq (432 \times 11)^2$, then we have

$$T_a(1) = \frac{\pi}{6}\frac{16a-1}{\sqrt{48a-1}} > 2\log a \geq \log a + 2\log 432Q > \log(432Qa). \tag{15}$$

Hence, by combining (13), (14), (15) and $C_Q \geq \mathrm{Max}\left\{2, (432Q)^2\right\}$, we can choose $a, b \geq C_Q$ to get the desired result. $\square$

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