Neural Bonanza III: The Final Bonanza Pt. 1

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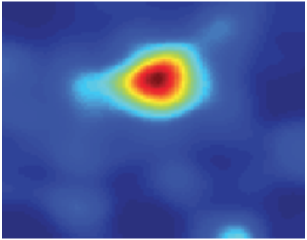
Outline

- Biological motivation
- Definitions
- Disproving conjectures
- Main question
- Future research

Biological Motivation

- Place cells in hippocampus
- Encode data
- Maps environment
- Convex place fields



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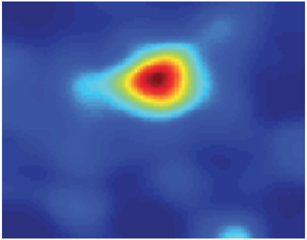


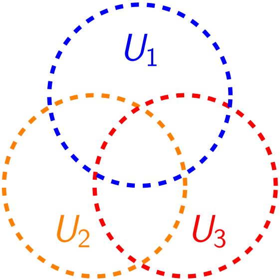
Relation to Mathematics

Can we find criteria to classify neural codes as convex given only the structure of the code?

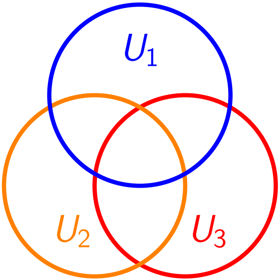
Important Definitions

Open/Closed Convex Codes

A code $C \subset 2^{[n]}$ is *open (or closed) convex* if there exist open (or closed) convex subsets $U_1, U_2, \ldots U_n \subseteq \mathbb{R}^d$, for some d, that generate the code.



Open Convex



Closed Convex

Important Definitions

3-Sparse

A code C is *3-sparse* if no codeword is longer than 3 neurons.

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Facet

A codeword $\sigma \in C$ is a *facet* if it is a maximal element of C with respect to inclusion, that is, $\sigma \nsubseteq \alpha$ for all $\alpha \in C$ such that $\alpha \neq \sigma$.

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Here, our facets are $\{123, 124, 34\}$

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A code C is *max-intersection complete* if all the intersections of its facets are in C.

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- $\{12, 3, 4\} \subseteq C$

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- $\{12, 3, 4\} \subseteq C$
- So C is max-intersection complete

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Simplicial Complex

We define the *simplicial complex* of a code C as:
$\Delta(C) := \{\sigma \subseteq [n] : \sigma \subseteq \alpha \text{ for some } \alpha \in C\}$

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Link

For a simplicial complex Δ and some $\sigma \in \Delta$, the *link* of σ is defined as: $\text{Lk}_\sigma(\Delta) := \{\tau \subseteq [n] \backslash \sigma : \sigma \cup \tau \in \Delta\}$

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Mandatory

A word $\sigma \in \Delta(C)$ is *mandatory* if $\mathrm{Lk}_\sigma(\Delta(C))$ is not contractible. Similarly, σ is *non-mandatory* if $\mathrm{Lk}_\sigma(\Delta(C))$ is contractible.

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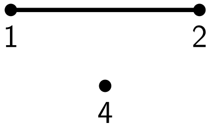
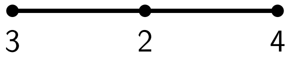


$\mathsf{Lk}_{\{1\}}(\Delta(C)) = \{23, 24, 2, 3, 4, \emptyset\}$



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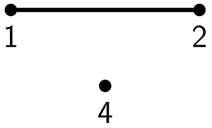

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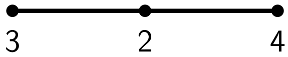


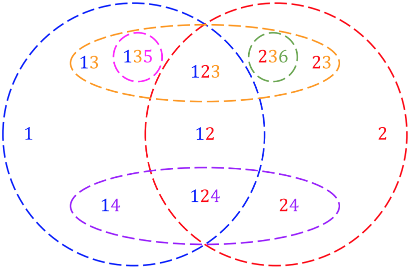
Locally Good

A code is *locally good* if it contains all of its mandatory codewords.

Disproven Conjectures: Goldrup and Phillipson

Conjecture (Goldrup and Phillipson 2014)

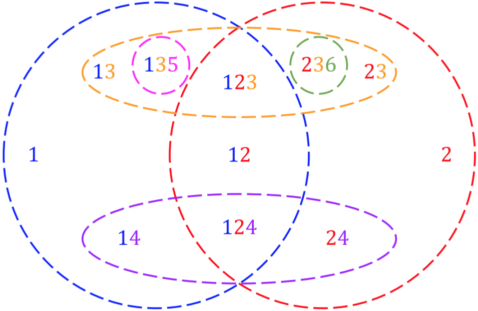
Let C be a code that is **open convex**, **not max intersection-complete**, and has at least **two non-mandatory codewords**. Suppose C has at least **3 facets** M_1, M_2, M_3, and there is $\sigma \in C$ such that $\sigma \subset M_1$ and $\sigma \cap M_2 \notin C$. Then C is not a closed convex code.



$$C = \{\underline{135}, \underline{123}, \underline{236}, \underline{124}, 12, 13, 14, 23, 24, 1, 2, \emptyset\}$$

Goldrup and Phillipson Conjecture

- Open convex



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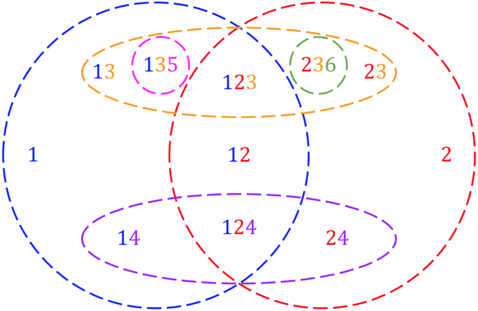

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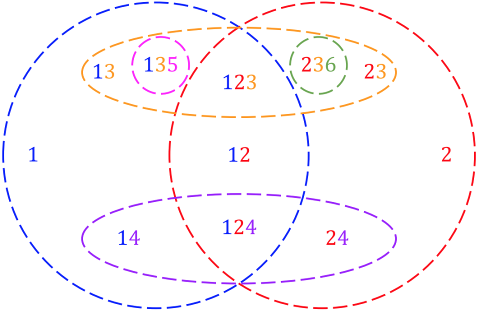
  - $135 \cap 236 = 3 \notin C$



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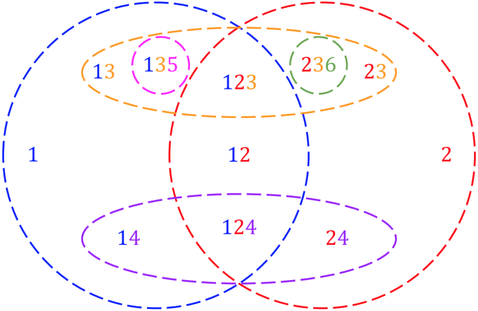
- ≥ 2 non-mandatory words



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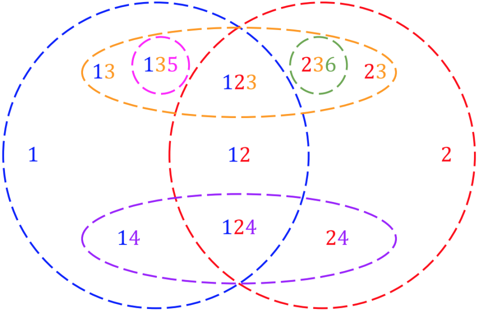

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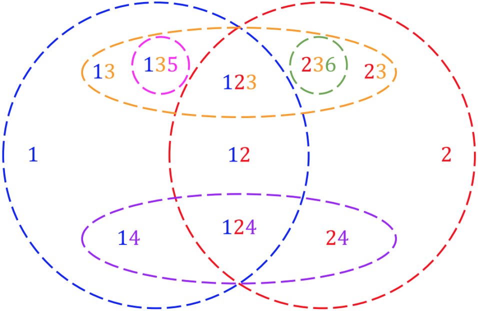

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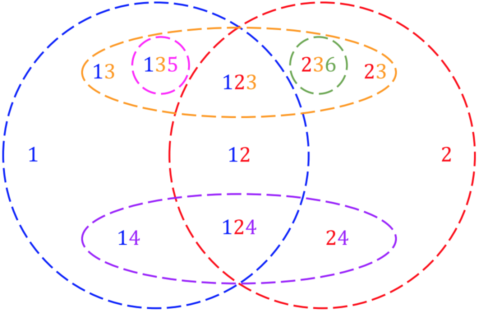
  - $M_1 = 123, M_2 = 236, M_3 = 135$



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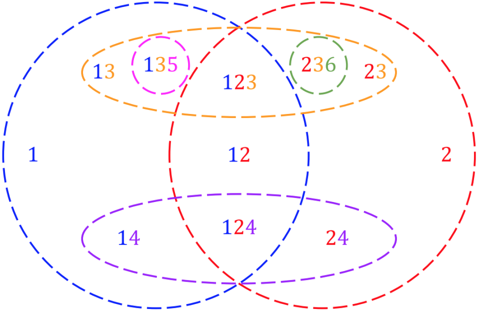
  - $M_1 = 123, M_2 = 236, M_3 = 135$
- $\sigma \in C$ such that:
  - $\sigma \subset M_1$.



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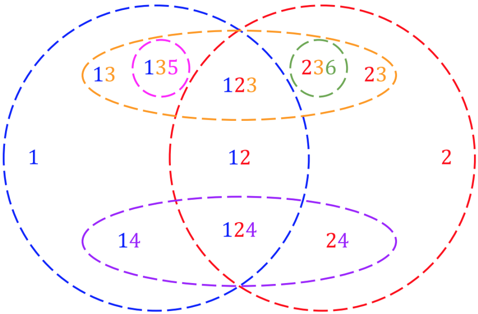
- $\sigma \in C$ such that:
  - $\sigma \subset M_1$. Let $\sigma = 13$. $13 \subset 123$



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- $\sigma \in C$ such that:
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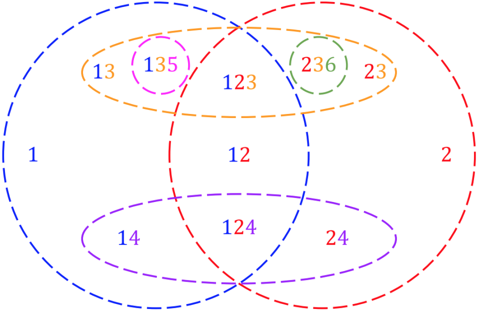

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  - $M_1 = 123, M_2 = 236, M_3 = 135$
- $\sigma \in C$ such that:
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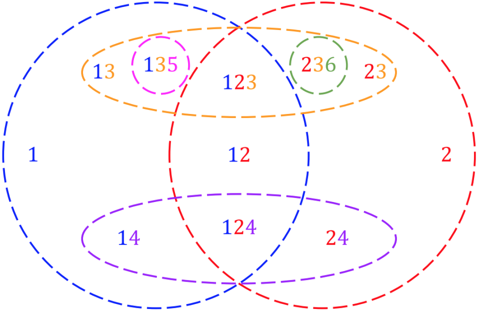
  - $\sigma \cap M_2 \notin C$. $13 \cap 236 = 3 \notin C$



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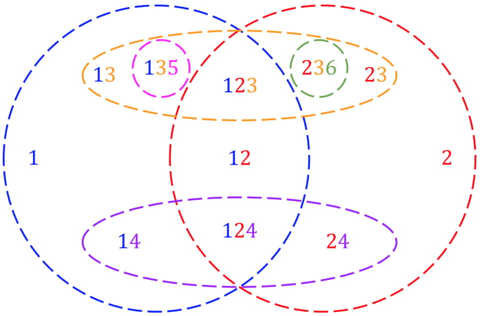

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Then the Conjecture says C is not closed convex...

Goldrup and Phillipson Conjecture

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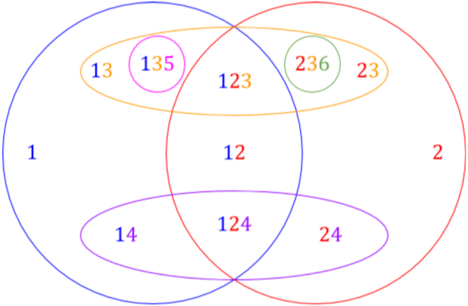

$\mathcal{C} = \{1, 13, 14, \underline{135}, \underline{123}, 12, \underline{124}, \underline{236}, 23, 24, 2, \emptyset\}$

Then the Conjecture says $\mathcal{C}$ is not closed convex... but this is **false!**

Main Question

What we already know:

- Convex $\Rightarrow$ locally good

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Conjecture 1

If a 3-sparse neural code is locally good, then it must be **closed** convex.

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What we already know:

- Convex $\Rightarrow$ locally good
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- But what about 3-sparse codes?

Conjecture 1

If a 3-sparse neural code is locally good, then it must be **closed** convex.

Conjecture 2

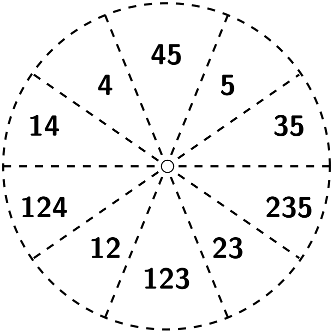
If a 3-sparse neural code is locally good, then it must be **open** convex.

Closed Convex

Conjecture 1

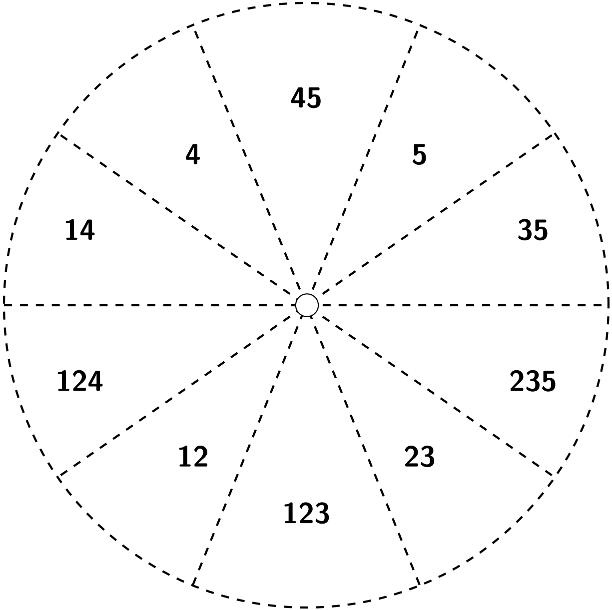
If a 3-sparse neural code is locally good, then it must be **closed** convex.

$$C = \{123, 124, 235, 12, 14, 23, 35, 45, 4, 5, \emptyset\}$$



Recall: Open convex $\Rightarrow$ locally good

Closed Convex



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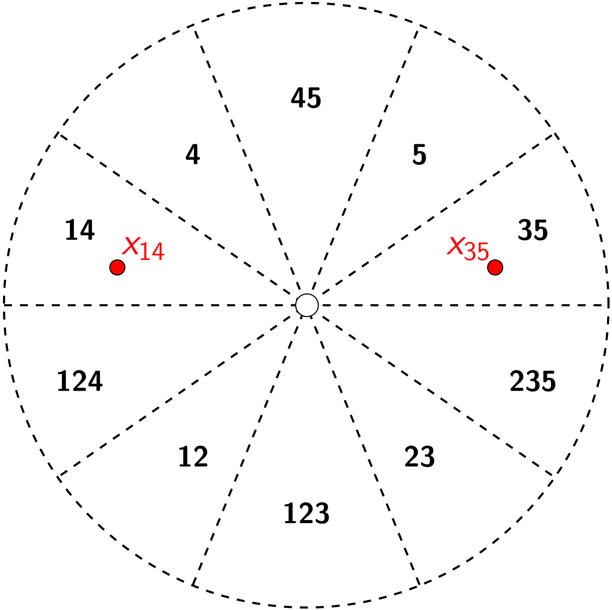

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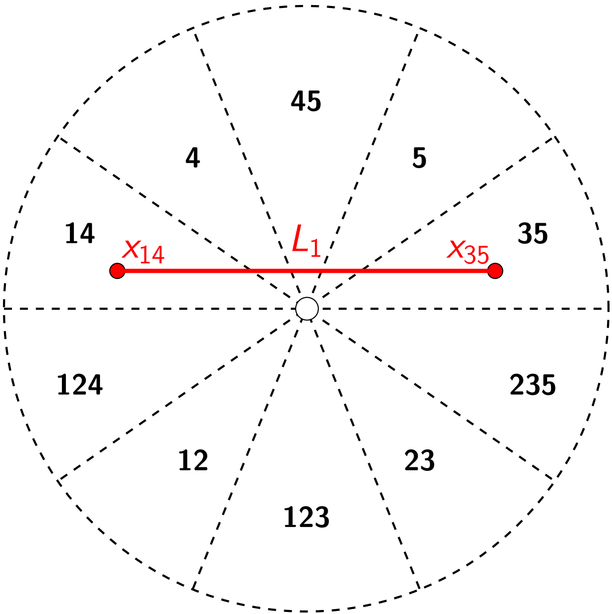

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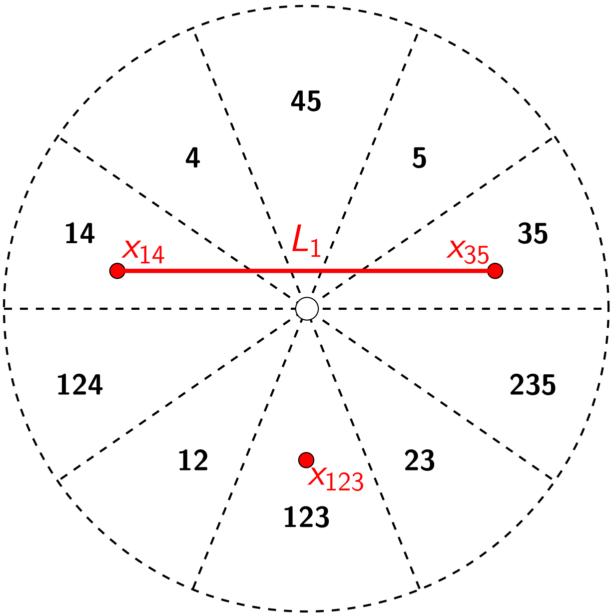

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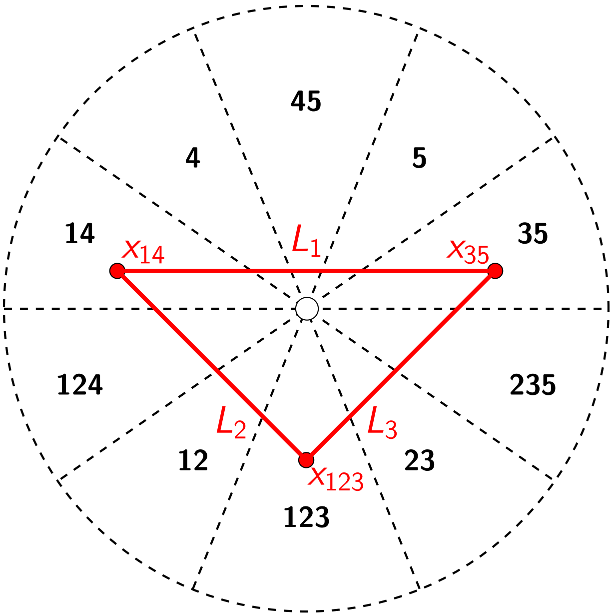

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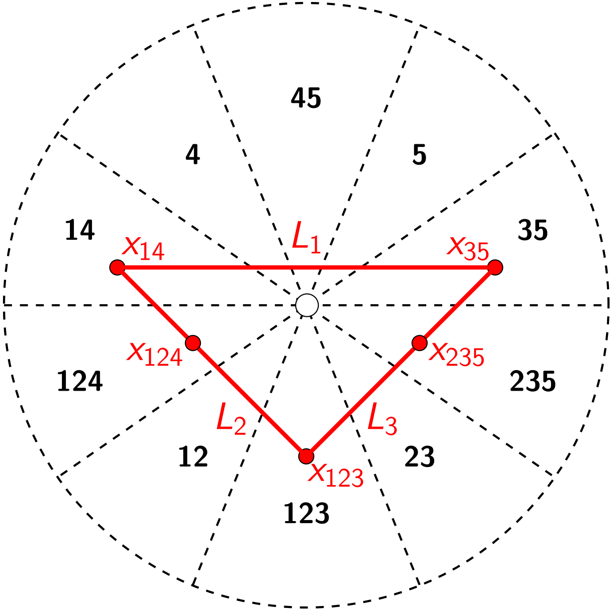

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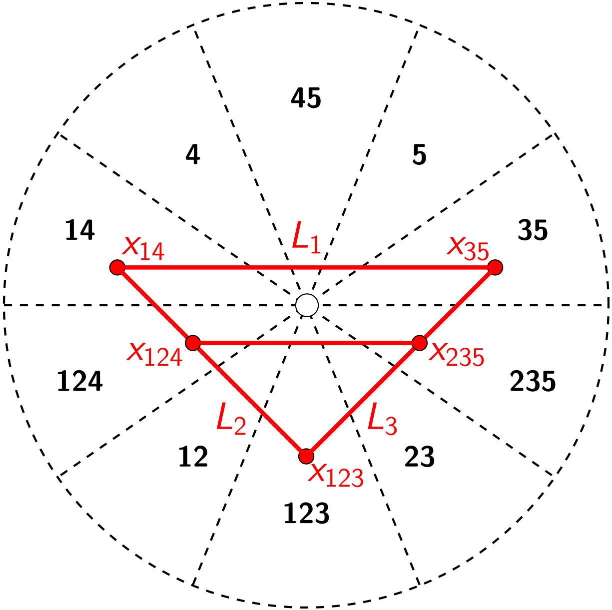
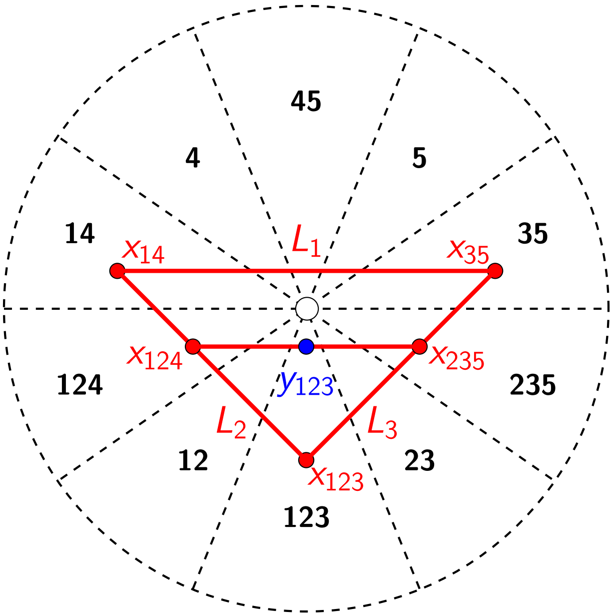


Closed Convex



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Theorem 4.3 (G. and Macdonald)

Let C be a neural code on n neurons with a closed convex cover $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$ that is fully dimensional. For $\sigma \subset [n]$, we define $U_\sigma = \cap_{i\in\sigma} U_i$. If there does not exist an $\alpha \in C$ such that U_α consists of a set that cannot be drawn in $\mathbb{R}^{d-1}$ or higher, then C is open convex.

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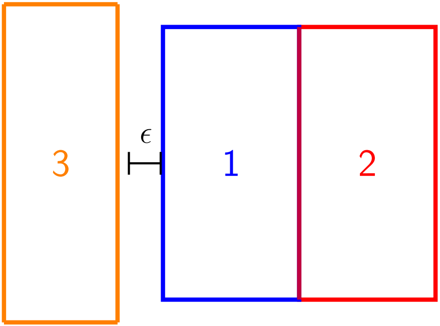

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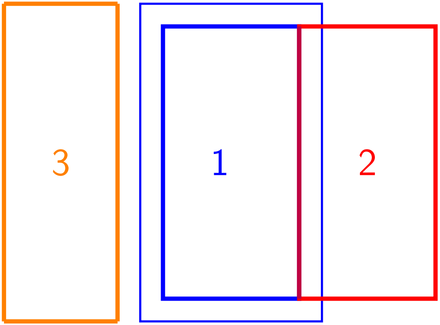

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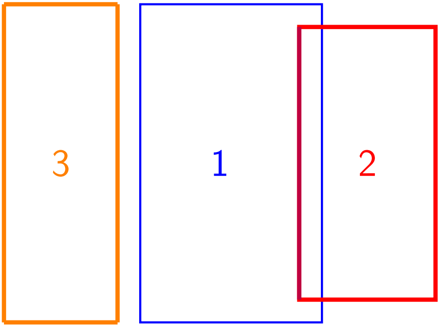

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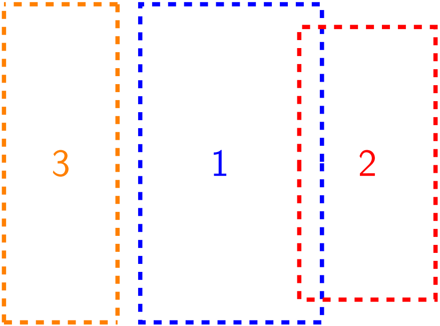

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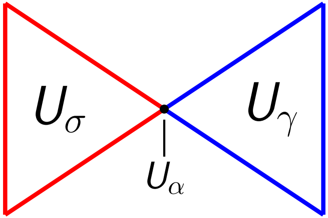
Let C be a neural code on n neurons with a closed convex cover $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$. If there exists a U_α that can only be expressed in $\mathbb{R}^{d-2}$ or below and is the intersection of exactly two sets in U, then C is open convex.



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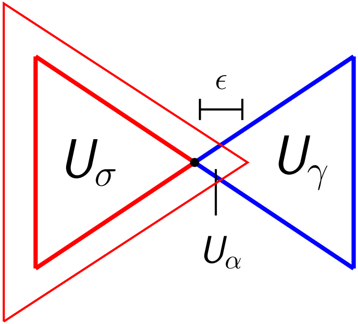

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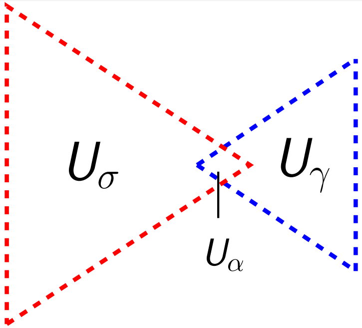

Possible Future Research

Conjecture

If C is a 3-sparse, locally good neural code on n neurons that is closed convex, then C is also open convex.

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Next Up

Find and define other criteria for open convexity that does not depend on closed convexity.

Thank You!

Thank you for listening!
Special thanks to Dr. Anne Shiu, Nida Obatake, Thomas Yahl, and the National Science Foundation.

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