NUMERICAL ANALYSIS QUALIFIER

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Do five problems, at least two from Problems 1–3 and two from 4–7.

Problem 1 Let $I = (-1, 1)$ and denote by P_n the space of polynomials of degree less than or equal to n. Let $\|f\|_\omega^2 = \int_I |f^2(x)|(1-x^2)^{-1/2}dx$ and define $\Pi_n f$ to be the best approximation to f in P_n in the norm $\|\cdot\|_\omega$, i.e.

$$\|\Pi_n f - f\|_\omega = \min_{\phi \in P_n} \|f - \phi\|_\omega.$$

(a) Give a formula for $\Pi_n f$ in terms of the Chebyshev polynomials $\{T_k(x)\}_{k=0}^n$.

(b) Let $f \in P_{n+1}$. Show that

$$\|f - \Pi_n f\|_\infty = \inf_{q \in P_n} \|f - q\|_\infty.$$

Problem 2 Consider the linear multistep method for $\frac{dy}{dt} = f(y, t)$

$$y_{n+1} - \sum_{j=0}^{k} a_j y_{n-j} = h \sum_{j=0}^{k} b_j f_{n-j}. \tag{2.1}$$

(a) Show that if (2.1) is consistent and $a_i > 0$ for $i = 0, \dots, k$, then (2.1) is stable.

(b) Consider $k = 1$. Show that there does not exist a stable method of the form of (2.1) which is of order 3.

(c) For $k = 1$, give an example of (2.1) which is of order 2 and satisfies $a_0 > 0, a_1 > 0$.

Problem 3 Consider the linear system of equations $Ax = b$ where $b \in \mathbf{R}^n$ and A is an $n \times n$ matrix given by

$$A = \begin{bmatrix} 2+h^2 & -1 & & & \\ -1 & 2+h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2+h^2 \end{bmatrix}, \quad \text{with} \quad h = 1/(n+1).$$

Denote by $\|x\|_A$ the energy norm, i.e., $\|x\|_A = \sqrt{x^T A x}$.

(a) Write down the Gauss-Seidel method for the linear system.

(b) Define the corresponding reduction matrix E and give an estimate for $\|E\|_\infty$.

(c) Derive an estimate for $\|E\|_A := \max_{x \neq 0} \|Ex\|_A / \|x\|_A$.

(d) How many iterations are sufficient to reduce the energy norm of the error by a factor of 10^{-6}?

Problem 4 Let $I = (0, 1)$ and denote by $\mathcal{T}_h$ a uniform partition of I with mesh size h. Denote by V_h the space of conforming linear finite elements defined over $\mathcal{T}_h$ and by Π_h the corresponding nodal value interpolation operator.

(a) Show that there exists a positive constant C, independent of h, such that

$$\|u - \Pi_h u\|_{L^2(I)} \le Ch^2 \|u\|_{H^2(I)}, \quad \forall u \in H^2(I)$$
$$|u - \Pi_h u|_{H^1(I)} \le Ch \|u\|_{H^2(I)}, \quad \forall u \in H^2(I).$$

(b) Prove that the orders of the approximation with respect to h cannot be improved.

Problem 5 Let $I = (0, 1)$. Consider the boundary value problem

$$u^{(4)}(x) = f(x), \qquad x \in I$$
$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

(a) Give a weak formulation of the boundary value problem of the form: Find $u \in V$ such that

$$a(u, v) = (f, v), \quad \forall v \in V.$$

Specify the energy space V and the bilinear form $a(u, v)$.

(b) Show that

$$\|v'(x)\|_{L^2(I)} \le \|v''(x)\|_{L^2(I)}, \quad \forall v(x) \in C^2(I) \quad \text{with} \quad v(0) = v(1) = 0.$$

Using this inequality show that the bilinear form $a(\cdot, \cdot)$ is coercive in V.

(c) Give a conforming finite element space for this problem.

Problem 6 Let $\hat{\tau}$ be the triangle with vertices $a_1(0, 0), a_2(1, 0), a_3(0, 1)$ and denote by $\{m_i\}_{i=1}^3$ the midpoints of the edges. Let $\{\phi_i\}_{i=1}^6$ be the nodal basis functions for quadratics corresponding to the nodes $\{a_i, m_i\}_{i=1}^3$.

(a) Show that $\{\phi_i\}_{i=1}^6$ form a unisolvent set.

(b) Define the element stiffness matrix for $\hat{\tau}$ corresponding to the Dirichlet form

$$D(u, v) = (\nabla u, \nabla v).$$

(c) Let $T(\hat{x}) = A\hat{x} + x_0$ be an affine map of $\hat{\tau}$ onto the arbitrary triangle τ (A is a 2×2 matrix). Express the stiffness matrix for τ in terms of that of $\hat{\tau}$ and A.

Problem 7 Let $\Omega \subset \mathbf{R}^2$ be a polygon. Given $f(t) = f(x,t)$ and $\phi(x)$ consider the weak formulation of the heat equation: find $u(t) = u(x,t) \in H_0^1(\Omega)$ (for each t) such that

$$(\dot{u}(t), v) + (\nabla u(t), \nabla v(t)) = (f(t), v), \quad \forall v \in H_0^1(\Omega)$$
$$u(0) = \phi(x).$$

Here f and ϕ are smooth functions and $\dot{u}(t) \equiv \frac{d}{dt}u(t)$.

(a) Let V_h be the space of conforming linear finite elements with respect to a fixed triangulation $\mathcal{T}_h$ of Ω. Define the semi-discrete Galerkin finite element approximation for the heat equation.

(b) Denote by $u_h(t) \in V_h$ the solution to the finite element problem. Show that there exists a constant C such that

$$\|u_h(t)\|_{L^2(\Omega)}^2 \leq \|u_h(0)\|_{L^2(\Omega)}^2 + C\int_0^t \|f(s)\|_{L^2(\Omega)}^2 ds.$$

(c) Let $T > 0$ be a fixed constant. Suppose that $\|u(t)\|_{H^2}$ and $\|\dot{u}(t)\|_{H^2}$ are bounded independent of $t \in [0, T]$. Show that there exists a constant $C > 0$, independent of h, such that

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} \leq Ch^2, \quad 0 \leq t \leq T.$$