NUMERICAL ANALYSIS QUALIFIER

May 1998

Problem 1. Let A be an $n \times n$ matrix.

(a) State a condition on the matrix A so that it has an LU factorization.
(b) Write down the LU factorization algorithm.
(c) Give an operation count for the algorithm.

NOTE: Do not interpret the question so that it becomes trivial (e.g. if A is the identity, then it has an LU factorization.)

Problem 2. Let $A \in \mathbf{R}^{n\times n}$ be a symmetric and positive definite matrix. Consider the linear system of equations

$$Ax = b$$

and the iterative method

$$x_{n+1} = x_n + \alpha_n r_n, \qquad r_n = b - Ax_n.$$

(a) Derive an expression for α_n such that the error $e_{n+1} = x - x_n$ satisfies

$$\|e_{n+1}\|_A = \min_{\alpha} \|e_n - \alpha r_n\|_A.$$

(b) Derive an estimate of the form

$$\|e_{n+1}\|_A \le \gamma \|e_n\|_A$$

with γ (strictly less than one) depending on the extreme eigenvalues of A.

Problem 3. Consider Gaussian quadrature applied to the problem

$$\int_a^b w(x) f(x)\, dx \approx \sum_{i=1}^n A_i f(x_i) \tag{1}$$

where $w(x) > 0$ almost everywhere on (a, b).

(a) Consider $a = -1$, $b = 1$, $w(x) = 1 - x^2$ and $n = 2$. Compute the nodes x_i and coefficients A_i which make (1) exact for all polynomials of degree 3.
(b) For the general case, show that $A_i > 0$ for $i = 1, \ldots, n$.
(c) Show that

$$\int_{-1}^1 w(x) f(x)\, dx - \sum_{i=1}^n A_i f(x_i) = \frac{f^{(2n)}(\zeta)}{(2n)!} \int_a^b w(x) q^2(x)\ dx$$

for some $\zeta \in (a, b)$. Here $q(x) = (x - x_1) \cdots (x - x_n)$.

HINT: You may use an error estimate for Hermite interpolation for part (c).

Problem 4. Consider the multi-step method

$$x_{n+1} = x_n + h(Af_n + Bf_{n-1} + Cf_{n-2}) \tag{2}$$

for the ordinary differential equation $x_t = f(x,t)$.

(a) Derive a linear system for the coefficients A, B, and C which makes the above multi-step method third order.

(b) Show that (2) produces the exact solution of $x_t = f(t)$ when f is a quadratic polynomial in t.

Problem 5. Let Ω be the unit square. Set Γ_0 to be the union of the boundary edges $y = 0$ and $y = 1$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. Consider the elliptic boundary value problem

$$\begin{aligned} -\Delta u + 2\nabla \cdot (bu) - (\nabla \cdot b)u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_0, \\ \frac{\partial u}{\partial n} - (b \cdot n)u &= 0 \text{ on } \Gamma_1. \end{aligned} \tag{3}$$

(a) Set up a variational formulation for the problem (3), i.e., define the energy space V and derive the weak formulation. (Hint: Integrate the first equation by parts in such a way that the boundary condition on Γ_1 becomes natural.)

(b) Prove or disprove: There exists a unique $u \in V$ satisfying the weak formulation.

(c) Given a finite element subspace $S_h \subseteq V$, define the Galerkin finite element approximation $u_h \in S_h$.

(d) Let $\{\phi_i\}_{i=1}^n$ be the nodal basis of S_h. Define the stiffness matrix M and give an algebraic equation which determines the coefficients of u_h in this basis.

Problem 6. This problem involves approximation to the one dimensional parabolic problem

$$u_t - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad x \in (0,1),\ t > 0$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \qquad u(x,0) = u_0(x), \quad x \in (0,1).$$

A two level finite difference approximation is given by

$$\frac{3}{2}w_j^n - 2w_j^{n-1} + \frac{1}{2}w_j^{n-2} = \frac{w_{j-1}^n - 2w_j^n + w_{j-1}^n}{h^2} + f_j^n$$

$$\text{for } j = 1, \ldots, m-1,\ n = 2, 3, \ldots,$$

$$w_0^n = w_m^n = 0, \qquad \text{for } n = 0, 1, \ldots,$$

$$w_j^0 = u_0(x_j), \qquad \text{for } j = 1, \ldots, m-1.$$

Here $h = 1/m$, $x_j = jh$, $t_n = nk$, $f_j^n = f(x_j, t_n)$, and w_j^n approximates $u(x_j, t_n)$. To be completely specified, this method requires generating the approximation $\{w_j^1\}$, for $j = 1, \dots, m-1$, by some other method, e.g. backward Euler.

(a) Compute the order of accuracy in time for this scheme.
(b) Determine under what conditions the method is stable.
(c) Assume that $f(x,t) = 0$. Show that in this case, the discrete solution w_j^n tends to zero (for every j) as n tends to infinity.

Problem 7. Let $f \in L^2(0,1)$. Consider the following two-point boundary value problem:

$$-u'' + u' = f \text{ in } (0,1)$$
$$u(0) = u(1) = 0.$$

(a) Derive a weak formulation for the two-point boundary value problem. (You need to identify the solution space.)
(b) Show that there is a unique solution for your weak formulation.
(c) Formulate the finite element problem with piecewise linear finite element space and state (without proof) an H^1-norm estimate for the error in the finite element approximation.
(d) Use part (c) to derive an L^2-norm estimate for the error in the finite element approximation.

HINT: You may assume (without proof) all the necessary regularity properties associate with this two-point boundary value problem.