Numerical Analysis Qualifier
May 1997

Attempt any five of the following problems.

You can use without proof the following Sobolev type inequalities in this exam:

$$\int_{\partial\Omega} |v|^2 dx \le C(\Omega) \int_{\Omega} (|\nabla v|^2 + |v|^2) dx \quad \text{and} \quad \int_{\Omega} |v|^2 dx \le C(\Omega) \left(\int_{\Omega} |\nabla v|^2 dx + \int_{\partial\Omega} |v|^2 dx \right).$$

Problem 1. Let A be a symmetric and positive definite operator on a finite dimensional inner product space V. Consider the following two-stage iteration,

$$\begin{aligned} x_{n+1/2} &= x_n + \tau_1(b - Ax_n), \\ x_{n+1} &= x_{n+1/2} + \tau_2(b - Ax_{n+1/2}). \end{aligned}$$

(a) Let $e_i = x - x_i$ where x is the solution of $Ax = b$. Compute the error propagation operator K satisfying

$$e_{n+1} = Ke_n.$$

(b) Knowing only that the spectrum of A lies in the interval $[\lambda_1, \lambda_2]$, give a criteria for the choice of τ_1 and τ_2 which gives rise to the most rapidly convergent iteration.

(c) Compute τ_1 and τ_2 (in terms of λ_1 and λ_2) satisfying (b).

Problem 2. Consider the Gaussian quadrature rule:

$$\int_a^b f(x)dx \approx \sum_{i=1}^n A_i f(x_i).$$

(a) Let $a = -1, b = 1$ and $n = 2$. Find the nodes x_i and weights A_i.

(b) Assume $f \in C^{2n}(a, b)$. Show that there exists $\xi \in (a, b)$ such that

$$E(f) \equiv \int_a^b f(x)dx - \sum_{i=1}^n A_i f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b \left[\prod_{i=1}^n (x - x_i) \right]^2 dx.$$

(c) Show that $A_i > 0$ for all $1 \le i \le n$ and all n.

Problem 3. Consider the initial value problem

$$\begin{cases} -(\kappa u')' + u = f(x), \\ u(0) = a, \quad u'(0) = \gamma, \end{cases} \tag{1}$$

where $\kappa(x)$ and $f(x)$ are sufficiently smooth functions.

(a) Write (1) as a system of two first order ordinary differential equations with unknowns u and $w = \kappa u'$.

(b) Consider a shooting method for solving the two point boundary value problem

$$\begin{cases} -(\kappa u')' + u = f(x) \\ u(0) = a, \quad u(1) = b. \end{cases}$$

Specifically, given a guess γ for $u'(0)$, consider the backward Euler method applied to the system in part (a) with uniform step size $h = 1/n$. This produces an approximate value for $u(1)$ which we denote by $\phi_n(\gamma)$. Show that there exists a positive constant N_0 such that if $n \geq N_0$ then

$$\phi_n(\gamma_1) \neq \phi_n(\gamma_2), \qquad \text{for} \quad \gamma_1 \neq \gamma_2.$$

(c) Suppose $\phi_n(\gamma_1)$ and $\phi_n(\gamma_2)$ have been computed for two distinct values $\gamma_1 \neq \gamma_2$. Show that

$$\gamma = \gamma_1 + \frac{b - \phi(\gamma_1)}{\phi(\gamma_2) - \phi(\gamma_1)}(\gamma_2 - \gamma_1)$$

satisfies

$$\phi_n(\gamma) = b$$

and hence solves the shooting problem.

Problem 4. Consider the boundary value problem

$$-u'' + q(x)u = f(x), \quad u(0) = u(1) = 0. \tag{2}$$

Here $q(x) \geq 0$ and $f(x)$ are sufficiently smooth functions.

(a) Describe the finite element method for this problem using piecewise linear elements; state the standard error estimates for $u - u_h$, where u and u_h are respectively the solutions to the boundary value problem (2) and the corresponding finite element approximation.

(b) Let $\{\phi_i\}$ be the standard nodal basis for the finite element space V_h and u_I be the finite element interpolant of u, i.e., $u_I(x) = \sum u(x_i)\phi_i(x)$. Show that

$$\int_0^1 (u' - u_I')v_h' dx = 0, \quad \text{for all} \quad v_h \in V_h.$$

(c) Use (b) to show that

$$\|u_h - u_I\|_{H^1(0,1)} \leq Ch^2 \|u\|_{H^2(0,1)}.$$

Problem 5. Consider the boundary value problem for a freely supported beam:

$$\begin{cases} u^{(4)} = f \quad x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

(a) State the variational formulation for this problem and specify the space V of test and trial functions.

(b) Show that the bilinear form is bounded and coercive in V (the norm in V is $\|\cdot\|_{H^2}$).

(c) State a H^2-norm estimates for the error in the case of cubic Hermite finite elements. Use this result and the duality argument to derive an estimate for the error in the L^2-norm. You can assume $\|u\|_4 \le C\|f\|_0$.

Problem 6. Let $\Omega = (0,1)^2$ and consider the elliptic boundary value problem

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u + \dfrac{\partial u}{\partial n} &= g \quad \text{in } \partial\Omega. \end{cases}$$

Here n denotes the outward normal direction. Assume that f and g are smooth functions.

(a) Derive the weak formulation for the Galerkin method. Specify the solution space V.

(b) Show that the corresponding bilinear form is bounded and coercive in V (with respect to $\|\cdot\|_{H^1}$ norm) and the linear form is bounded in V.

(c) Now partition the domain Ω into **two triangles** by connecting its lower-left and upper-right corners. Let V_h be the space of continuous functions which are piecewise linear with respect to this partition. Compute the stiffness matrix.

Problem 7. Consider the following parabolic initial-boundary value problem

$$\begin{cases} u_t = u_{xx} + f(x,t), \quad x \in (0,1), t \in (0,\infty) \\ u(x,0) = u^0(x),\ x \in (0,1) \\ u(0,t) = u(1,t) = 0,\ t \in (0,\infty). \end{cases}$$

Denote by Δx and Δt the step sizes in the x and t direction and assume that $\Delta x = 1/M$. Consider the finite difference approximation of this problem on the mesh $(x_i, t_n) = (i\Delta x, n\Delta t), i = 0, \dots, M, n = 0, \dots$

$$\begin{cases} \dfrac{y_i^{n+1} - y_i^n}{\Delta t} = \dfrac{y_{i+1}^{n+1} - 2y_i^{n+1} + y_{i-1}^{n+1}}{\Delta x^2} + f_i^{n+1}, \quad i = 1, \dots, M-1,\ n = 0, 1, \dots, \\ y_0^n = y_M^n = 0,\ n = 0, 1, \dots, \\ y_i^0 = u^0(x_i), \quad f_i^n = f(x_i, t_n). \end{cases}$$

Here y_i^n is an approximate for $u(x_i, t_n)$.

(a) Define the local truncation error and estimate it in terms of Δx and Δt.

(b) Show that the scheme is unconditionally stable.

(c) State the L^2-norm estimate for the error.