Numerical Analysis Qualifier
Spring 1996

1. Attempt any five of the following problems. Each problem is 20 points. The points in the subproblems indicate the relative weight of that part.
2. Start each problem on a new sheet of paper and write on one side of each sheet of paper.

Problem 1.

1. (3 pts) For $x \in R^n$ let $\|x\|$ denote its vector norm. For an $n \times n$ real matrix A introduce the norm which is subordinate to (induced by) the vector norm $\|\cdot\|$.
2. (6 pts) Define the condition number $cond(A)$ for a nonsingular $n \times n$ real matrix A, which is related to the above introduced matrix norm. Show that if A is symmetric and $||.||$ is the Euclidean norm in R^n then $cond(A) = \max_i |\lambda_i| / \min_i |\lambda_i|$, where λ_i are the eigenvalues of A.
3. (11 pts) Compute the condition number of the following matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

Show the main steps of your reasoning. If you have difficulty in computing $cond(A)$ exactly, try to make an estimate from above.

Problem 2. Consider the $n \times n$ symmetric and positive definite matrix A and the $k \times n$ matrix B as operators:

$$A : R^n \longrightarrow R^n, \qquad B : \; R^n \longrightarrow R^k, \qquad B^T : \; R^k \longrightarrow R^n.$$

Let $f \in R^n$ and $g \in R^k$ be given and assume that $BA^{-1}B^T$ is nonsingular.

1. (5 pts) Show that the linear system

$$Au + B^T p = f, \qquad Bu = g$$

has a unique solution (u, p).

2. (10 pts) Define a functional $J: \; R^n \times R^k \longrightarrow R$ as follows:

$$J(v,q) = \frac{1}{2}(Av,v) + (B^T q, v) - (f,v) - (g,q).$$

Show that

$$J(v,q) = \frac{1}{2}(A^{-1}(Av+B^Tq-f), Av+B^Tq-f) - \frac{1}{2}(A^{-1}B^T(p-q), B^T(p-q)) + J(u,p).$$

3. (5 pts) Use this to prove that (u,p) is a saddle point of $J(v,q)$: i.e.

$$\sup_{q\in R^k} \inf_{v\in R^n} J(v,q) = J(u,p).$$

Problem 3.

1. (8 pts) Let $p_k(x)$ be the polynomial of degree k which interpolates the function f at the distinct points $x_0, x_1, ..., x_k$, for arbitrary k. Define

$$c_k = \frac{1}{k!}\frac{d^k p_k(x)}{dx^k} = \frac{p_k^{(k)}(x)}{k!},$$

and let

$$g_k(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + ... + c_k \prod_{j=0}^{k-1}(x-x_j)$$

Prove that $g_k(x) = p_k(x)$.

2. (7 pts) The formula

$$p_k(x) = \sum_{j=0}^{k} c_j \prod_{i=0}^{j-1}(x-x_i) = \sum_{j=0}^{k} \frac{p_j^{(j)}(x)}{j!} \prod_{i=0}^{j-1}(x-x_i)$$

is not constructive. Give a constructive algorithm for computing the c_j's.

3. (5 pts) If f has $n+1$ continuous derivatives, find an upper bound for the interpolation error in terms of $f^{(n+1)}$. Justify your answer.

Problem 4. Given the initial value problem $y' = f(x,y), \quad y(x_0) = y_0$:

1. (10 pts) Derive an explicit multistep (Adams-Bashforth) method of order 3 that uses integration over the interval (x_{i-2}, x_{i+1}). Check whether the corresponding linear difference equation satisfies the stability condition.

2. (10 pts) Write a multistep method for this problem which is consistent but does not satisfy the stability condition. Show that your method is consistent but does not satisfy the stability condition.

Problem 5. Given the b.v.p.:

$$-\Delta u + u = f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + u = g \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in R^2 with Lipschitz boundary $\partial\Omega$ and n is the outward normal unit vector to $\partial\Omega$.

1. (4 pts) Derive the variational formulation to this problem. Describe the energy space.

2. (8 pts) Let Ω be partitioned into triangular finite elements. In each triangle given by its vertices a^i, $i = 1, 2, 3$, introduce the barycentric (homogeneous or area) coordinates λ_i, $i = 1, 2, 3$. Consider the following finite element space V_h on this partition: the restriction of a function in V_h over each finite element T is a quadratic polynomial augmented with a cubic "bubble" $\lambda_1\lambda_2\lambda_3$; the degrees of freedom are the values at the vertices, the midpoints of the edges, and the barycenter of T. Find a nodal basis for V_h.

3. (8 pts) Now consider the finite element space which is a slight modification of the space of the previous subproblem by changing the set of degrees of freedom: instead of the value at the barycenter we take the mean value of the function over the finite element T. Find a nodal basis for V_h. Hint: For computing the corresponding integrals over T use the Bell-Holland formula.

Problem 6. The finite element method with piecewise linear basis functions is to be used to approximate the solution to the problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma_1, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_2.$$

The domain Ω is the square with vertices at (0,0), (1,0), (0,1), and (1,1). The set Γ_1 is the part of the boundary of Ω on the line $x = 1$, while Γ_2 is the rest of the boundary. The derivative $\frac{\partial u}{\partial n}$ is in the direction of the exterior normal. The domain is triangulated by first making a grid of n^2 uniform squares with sides of length $h = 1/n$, and then bisecting each square with a diagonal parallel to the line $y = x$. The active nodes are numbered left to right, bottom row to top row.

1. (5 pts) Give the variational formulation of this problem and then outline the steps in the finite element method.

2. (7 pts) Explain what is meant by the term "local stiffness matrix". Find the local stiffness matrix for one finite element.

3. (8 pts) Explain what is meant by the term "global stiffness matrix". Find the first two rows of the global stiffness matrix for this problem.

Problem 7. Consider the initial problem for the transport equation:

$$\begin{aligned} u_t + bu_x &= F \quad \text{for } -\infty < x < \infty,\, t > 0, \\ u(x,0) &= f \quad \text{for } -\infty < x < \infty, \end{aligned}$$

where $b > 0$ is a given constant and $F(x,t)$ and $f(x)$ are given smooth functions. Consider the following modified Lax-Friedrichs scheme

$$\frac{1}{\Delta t}\{\frac{1}{2}(U_{i+1}^{n+1} + U_{i-1}^{n+1}) - U_i^n\} + \frac{b}{2h}(U_{i+1}^{n+1} - U_{i-1}^{n+1}) = F_i^{n+1}, \quad U_i^0 = f(x_i),$$

on the grid $\{(x_i, t_n) :\ x_i = ih,\ t_n = n\Delta t,\ i = 0, \pm 1, ...,\ n = 0, 1, ...\}$. Here Δt and h are the time and space mesh step sizes, and U_j^n is an approximation to $u(x_j, t_n)$.

1. (5 pts) Estimate the local truncation error of this approximation.

2. (8 pts) Although the above scheme may not be computationally practical, under the assumption that its solution exists, show that if the step-sizes Δt and h satisfy $\frac{b\Delta t}{h} \geq 1$ then the scheme is stable in maximum norm.

3. (7 pts) Now consider this equation for $0 < x < 1,\ t > 0$ and add the boundary condition at $x = 0$: $u(0,t) = 0$ for $t > 0$ assuming compatibility of the initial and boundary conditions, i.e. $f(0) = 0$. Derive a consistent approximation to this boundary condition. Add this approximation to the above scheme. Write the formulas for U_0^1, U_1^1, U_2^1 and U_3^1.