Numerical Analysis Qualifier
Spring 1995

1. Attempt any five of the following problems. Each problem is 20 points. The points in the subproblems indicate the relative weight of that part.

2. Start each problem on a new sheet of paper and write on one side of each sheet of paper.

Problem 1. Given the data:

x	0	1/3	1
f(x)	0	1/2	1

Find a cubic spline that interpolates these data and satisfies the natural boundary conditions. Determine the approximate value of $f(\frac{1}{4})$ using the spline.

Problem 2. Let $Q_0,\ Q_1, ...,\ Q_n$ be the set of orthogonal polynomials on $(0, 1)$ with weight $p(x) = x$, where Q_k has degree k and leading term x^k.

(a) (5 pts) Find $Q_0,\ Q_1,\ \mathit{and}\ \ Q_2$ of this sequence;

(b) (5 pts) Construct an interpolatory quadrature formula for evaluating $\int_0^1 xf(x)dx$ that uses the root x_1 of Q_1 as a quadrature node. Check the algebraic degree of accuracy (precision) of this formula;

(c) (10 pts) Show that the Gaussian quadrature formula for $\int_0^1 xf(x)dx$ that uses the roots of Q_n as quadrature nodes has algebraic degree of accuracy $2n - 1$. Prove that the coefficients of this formula are positive.

Problem 3. Let A be a real $n \times n$ symmetric matrix with nonzero diagonal elements.

(a) (5 pts) Formulate the SOR iteration for the system $Ax = b$;

(b) (6 pts) Show that if the SOR iteration parameter ω satisfies the the inequality $|\omega - 1| \geq 1$ then the iteration may diverge;

(c) (9 pts) Prove that if A is positive definite then the SOR iteration converges for $0 < \omega < 2$. Hint. Show that the matrix norm of the error propagation matrix for SOR that is induced by (or subordinate to) the vector norm $||x||_A^2 = (Ax, x)$ is strictly less than 1.

Problem 4. Given the initial value problem $y' = f(x, y), y(x_0) = y_0$:

(a) (10 pts) Derive an explicit multistep (Adams-Bashforth) method of order 3 that uses integration over the interval (x_{i-1}, x_{i+1});

(b) (7 pts) Derive an estimate for the local truncation error;

(c) (3 pts) Check whether the corresponding linear difference equation satisfies the stability condition.

Problem 5. Consider the two-point boundary value problem

$$-u''(x) + u(x) = f(x), \ 0 < x < 1, \tag{1}$$

$$u'(0) = 0 = u'(1), \tag{2}$$

where $f \in C^1[0, 1]$. Let V_h be the set of continuous, piecewise quadratic polynomials with respect to a uniform partition of the interval [0,1]. Let $u_h \in V_h$ be the finite element approximation to the true solution u of (1), (2).

(a) (4 points) Give the variational form of the problem (1), (2). What is the energy norm for the problem?

(b) (7 points) Prove the following error estimate: There is a positive constant, C, such that

$$\|u - u_h\|_{H^1(0,1)} \leq C\|u - v_h\|_{H^1(0,1)} \text{ for every } v_h \in V_h;$$

(c) (3 points) State estimates for $\|u - u_h\|_{L_2(0,1)}$ and $\|u - u_h\|_{H^1(0,1)}$ in terms of h, the grid size of the partition;

(d) (6 points) Give the variational form and energy norm for the following problem, and prove the statement in part b, with the set V_h changed to the set of continuous, piecewise quadratics which satisfy boundary condition (2′) below:

$$-u''(x) \ = f(x), \ 0 < x < 1, \tag{1'}$$

$$u(0) = 0 = u(1). \tag{2'}$$

Problem 6. Let K be the triangle in the x, y plane, with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Let $P_r(K)$ be the set of polynomials of degree at most r defined for x and y in K.

(a) (5 points) Find a nodal basis for $P_1(K)$ with respect to the nodes $N_1 = (0,0)$, $N_2 = (1,0)$, $N_3 = (0,1)$; that is, find functions $\lambda_i \in P_1(K)$, $i = 1,2,3$, satisfying $\lambda_i(N_j) = \delta_{ij}$;

(b) (8 points) Find a nodal basis $\{\phi_i\}_{i=1}^{6}$ for $P_2(K)$ with respect to the six nodes consisting of the vertices of K and the midpoints of the edge s, numbered counterclockwise stating at $(0,0)$;

(c) (7 points) Find a nodal basis $\{\psi_i\}_{i=1}^{10}$ for $P_3(K)$ with respect to the nodes consisting of the 10 points in K having x and y coordinates either 0, 1/3, 2/3, or 1. Number the first nine nodes counterclockwise starting at (0,0), and let node 10 be the point with coordinates (1/3,1/3).

Problem 7. Consider the two-level difference scheme for the 1-D heat equation $u_t - u_{xx} = 0$:

$$\frac{U_j^{n+1} - U_j^n}{k} = \sigma \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} + (1-\sigma)\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2},$$

where k and h are the time and space mesh step sizes, σ is a real parameter, and U_j^n is approximation for $u(x_j, t_n)$, $x_j = jh$, $t_n = nk$.

(a) (7 pts) Estimate the local truncation error of this approximation. What is the local truncation error for $\sigma = 0.5$?

(b) (10 pts) Apply the simplest Fourier mode analysis to find those σ for which the scheme is unconditionally stable. (Here you either consider unbounded spatial domain, of simply disregard the possible boundary conditions);

(c) (3 pts) Is this scheme stable for $\sigma = 0, \quad 0.5, \quad 1$? Explain.