Numerical Analysis Qualifier
Spring 1994

1. Attempt any five of the following problems. Each problem is 20 points.
2. Start each problem on a new sheet of paper and write on one side of each sheet of paper.

Good luck!

Problem 1.

1. (10 pts) Prove the min max theorem for the eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ of a symmetric matrix $A \in R^{n \times n}$:

$$\lambda_i = \min_{dim S = i} \max_{x \in S} \frac{x^T A x}{x^T x}.$$

2. (10 pts) Show that

$$\lambda_n = \max_{x \in R^n} \frac{x^T A x}{x^T x}$$

and use this result to show that the following matrix has an eigenvalue greater than 67.4:

$$\begin{bmatrix} 2 & 4 & 5 & 1 \\ 4 & 2 & 1 & 3 \\ 5 & 1 & 60 & 12 \\ 1 & 3 & 12 & 48 \end{bmatrix}.$$

Problem 2.

1. (10 pts) Define the condition number of a matrix A and explain in detail how it is related to the numerical solving of the system $Ax = b$. Prove that $||A||||A^{-1}|| \geq 1$ for any operator norm $||.||$, provided A^{-1} exists.

2. (10 pts) Let $A \in R^{n \times n}$ be a nonsingular matrix. Prove that the condition number $\kappa(A)$ satisfies

$$\kappa(A) \geq \frac{||A||}{||A - B||}$$

for all singular matrices $B \in R^{n \times n}$. Hint: Use the fact that for any singular matrix B there is a nonzero vector z, such that $Bz = 0$.

Problem 3. Let $P(x)$ be the polynomial of lowest degree that interpolates the data:

1	2	4
$f(1)$	$f(2)$	$f(4)$
$f'(1)$		$f'(4)$

1. (5 pts) What is the degree of $P(x)$? Is this polynomial unique ? (Justify your answer.)
2. (10 pts) If $f(x) = \ln x$ show that $P(x) \leq \ln x$ in the interval $(2, 4)$.
3. (5 pts) Find $P(x)$ for $f(x) = \ln x$. Note: $\ln 2 = 0.693147$.

Problem 4.

1. (5 pts) Derive a formula of numerical differentiation of the form:

$$f''(x) \simeq Af(x) + Bf(x+h) + Cf(x+2h) \equiv S(x,h)$$

of as high order of approximation as possible.

2. (5 pts) Use the Taylor expansion to derive an expression for the error of the above approximation.
3. (10 pts) Use Richardson's extrapolation with $S(x,h)$ and $S(x,h/2)$ to obtain a higher order approximation. What can you say about the error ?

Problem 5 . Consider the quadrature rule for evaluating $\int_{-2}^{2} f(x)dx$:

$$I(f) = Pf(0) + Q(f(-1) + f(1)).$$

1. (5 pts) Determine the weights P and Q in such a way that this quadrature has the highest possible algebraic degree of accuracy;
2. (10 pts) Give the appropriate Peano kernel for the error of this quadrature. Show that the error $R(f)$ can be expressed in the form:

$$R(f) \equiv I(f) - \int_{-2}^{2} f(x)dx = Af^{(4)}(\xi), \text{ for some } \xi \in (-2, 2).$$

3. (5 pts) Determine the constant A from (2).

Problem 6. The boundary value problem

$$-u'' + u = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0$$

is approximated by the finite difference scheme:

$$-\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + (1 + \frac{h^2}{12})y_i = \frac{1}{12}(f_{i-1} + 10f_i + f_{i+1}) + \frac{h^2}{12}f_i \equiv F_i,$$

for $\quad i = 1, ..., n-1$, and $\quad y_0 = y_n = 0$,

where $h = 1/n$ and y_i is an approximation to $u_i = u(x_i), \quad x_i = ih, \quad i = 0, 1, ..., n$.

1. (5 pts) Show that the local truncation error of this scheme is $O(h^4)$, provided that $f(x)$ is a sufficiently smooth function.

2. (10 pts) Using matrix stability analysis show that the solution of this scheme satisfies the *a priori* estimate:

$$\sum_{i=1}^{n-1} y_i^2 h \le \sum_{i=1}^{n-1} F_i^2 h.$$

3. (5 pts) Using either a maximum principle or an energy type of analysis, show that the error of the method $e_i = y_i - u(x_i)$ satisfies the estimate

$$\max_{0\le i\le n} |e_i| \le Ch^4$$

with constant C, independent of h.

Problem 7. The function $u(x, y, t)$ is a solution to the following problem for the heat equation:

$$u_t - u_{xx} - u_{yy} = f(x, y, t), \text{ for } (x, y) \in \Omega = (0, 1)^2, t > 0,$$

$$u(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(1, y, t) = 1, \quad 0 < y < 1, \quad t > 0,$$
$$u(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, 1, t) = 1, \quad 0 < x < 1, \quad t > 0,$$
$$\text{and } u(x, y, 0) = 2, \quad in \quad \Omega,$$

where $f(x, y, t)$ is a given function.

1. (5 pts) Using the finite volume method on a uniform square grid derive an explicit finite difference approximation to this problem.

2. (15 pts) Apply Fourier analysis to find the exact Courant condition for the stability of this scheme in L^2-norm.

Problem 8. Given the boundary value problem for $u(x, y)$:

$$-\Delta u \equiv -u_{xx} - u_{yy} = 1, \quad in \quad \Omega = (0, 2)^2,$$

$$\frac{\partial u}{\partial x}(2, y) = y(2 - y) \text{ for } 0 < y < 2,$$

and $u = 0$ on the remaining part of the boundary of the domain Ω.

1. (5 pts) Give the weak formulation to this problem. Let S^h be a finite element subspace of the corresponding energy space. Write down the Ritz-Galerkin system for the approximate solution to this problem in S^h. For S^h being the subspace of piece-wise polynomial functions of degree k state the error estimates for the finite element solution in H^1- and L^2-norms.

2. (15 pts) The square Ω is divided into 8 equal triangles and S^h is the space of the linear triangular finite elements based on this triangulation. Find the approximation of u in S^h.

Problem 9. The modified Euler's method (also called a second order Runge-Kutta method, or Huen's method) for the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad t > t_0, \quad y(t_0) = y_0,$$

is

$$u_{n+1} = u_n + h(k_1 + k_2)/2, \quad u_0 = y_0, \quad t_{n+1} = t_n + h,$$

where $k_1 = f(t_n, u_n)$, $k_2 = f(t_n + h, u_n + hk_1)$, and u_n approximates $y(t_n) = y_n$.

1. (8 pts) Use Taylor's theorem to obtain an expression for the local truncation error of the method.

2. (12 pts) Use the above result to devise a scheme for adaptive step-size control. Specifically, compare the result of one step of size h with the result of two steps of size $h/2$ to estimate the error of one step. The objective of the method shoud be to adjust the step-size h, as efficiently as possible, to ensure that the error in advancing the solution from t_n to $t_n + h$ is at most ϵh, where ϵ is a prescribed error tolerance.