NUMERICAL ANALYSIS QUALIFIER

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Solve as many problems as you can. Show all of your work.

Problem 1. (a) Consider the quadrature problem

$$I(f) \approx \int_0^\infty e^{-x}\ f(x)\ dx.$$

Show that

$$Q_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

for $n = 1, 2, \ldots$ is a polynomial of degree n which is e^{-x}-orthogonal to P_{n-1} on $[0, \infty)$.

(b) Compute the third order Gaussian quadrature approximation using the results from (a).

(c) Give an expression for an n'th order nontrivial polynomial $Q_n(x)$ satisfying

$$\int_{-\infty}^{\infty} e^{-x^2}\ Q_n(x)\ p(x)\ dx = 0$$

for all polynomials $p \in P_{n-1}$.

Problem 2. Let $A = D - L - U$ be a symmetric and positive definite $n \times n$ matrix. Here D is a diagonal matrix and U and L are strictly upper and lower triangular matrices, respectively.

(a) For $\omega \in (0, 2)$, define the successive over relaxation iteration (SOR(ω)) and its symmetrization, i.e., the symmetric successive over relaxation iteration (SSOR(ω)), for iteratively computing the solution of $Ax = b$ ($x, b \in R^n$).

(b) Let G denote the reduction matrix associated with the SSOR(ω) iteration. Show that G can be written $I - BA$ by explicitly computing B in terms of ω, L, D and U. Here I is the $n \times n$ identity matrix and B is an $n \times n$ matrix.

(c) Let $\tilde{G}$ be the reduction matrix associated with SOR(ω) and assume that $\|\tilde{G}\|_A \leq \rho$. Here $\|\cdot\|_A$ is the matrix norm that is subordinate to the vector norm $\|x\|_A = (Ax, x)^{1/2}$ for $x \in R^n$. Show that the condition number $K(BA) \equiv \|BA\|_A \|(BA)^{-1}\|_A$ is bounded by $(1 - \rho^2)^{-1}$.

Problem 3. Consider the initial boundary value problem for the one dimensional heat equation

$$\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \qquad 0 < x < 1,\ t > 0,\\
u(0, t) = 0,\ \frac{\partial u}{\partial x}(1, t) &= 0, \qquad t > 0,\\
u(x, 0) &= f(x), \quad 0 < x < 1.
\end{aligned}$$

Consider the Crank-Nicolson finite difference approximation

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{2h^2} + \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{2h^2},$$

$$j = 1, 2, ..., N-1,\ n = 0, 1, ...$$

where k and h are the time and space mesh step sizes, and U_j^n is approximation for $u(x_j, t_n)$, $x_j = jh$ for $j = 1, 2, ..., N-1$, $Nh = 1$, and $t_n = nk$ for $n = 0, 1, ...$

(a) Estimate the local truncation error of this approximation for all internal points assuming that the solution is as smooth as needed by your analysis.
(b) Use $U_0^n = 0$, $n = 0, 1, ...$ for the left boundary condition, and $U_j^0 = f(x_j)$, $j = 1, 2, ..., N-1$ as the initial condition. Derive a second order approximation for the boundary condition at $x = 1$ that will lead to a symmetric system for the unknowns U_j^{n+1}, $j = 1, \dots, N$.
(c) Prove that the scheme is unconditionally stable in a discrete L^2-norm.

Problem 4. Given the initial value problem for the hyperbolic equation in one dimension: Find $u(x,t)$ such that:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= F(x,t), \quad 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) &= 0, \quad t > 0, \\ u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) &= g(x), \quad 0 < x < 1. \end{aligned}$$

(a) Derive the weak formulation and its semi-discrete Galerkin approximation using piecewise linear finite elements on a uniform spatial mesh $x_i = ih$, $i = 0, 1, \dots, N$ with $h = 1/N$. What would be the appropriate energy type *a priori* estimate for the semi-discrete Galerkin solution?
(b) Derive a fully discrete explicit approximation on a grid $t_n = nk$ by replacing the time derivative by the three-point finite difference in time. Using the Fourier method show that this scheme is stable if $k \le h$. Here k and h are, respectively, the step-sizes in time and space.

Problem 5. Let Ω be a convex polygonal domain in $\mathbf{R}^2$ and let $\mathcal{T}_h$ be a triangulation of Ω of size h. Consider the P^1-conforming finite element space V_h, i.e. the set of functions which are piecewise linear with respect to the triangles of $\mathcal{T}_h$ and continuous in Ω. This space is used to approximate the solution of the problem

$$-\Delta u + u = f \ \text{ in } \ \Omega, \qquad \frac{\partial u}{\partial n} = g \ \text{ on } \ \partial\Omega.$$

Here n is the outer unit vector, normal to $\partial\Omega$, and f and g are sufficiently smooth functions defined on Ω and $\partial\Omega$, respectively. Assume that the solution of this problem is H^2-regular for $f \in L^2(\Omega)$ and g sufficiently smooth.

(a) Derive the weak formulation of this problem. Show that the corresponding linear form is bounded on $H^1(\Omega)$.
(b) Let u_h be the finite element approximation of the above problem which uses a trapezoidal quadrature rule for computing the integrals involving the function g. Estimate the error of the method in the $H^1(\Omega)$-norm assuming that the function g is as smooth as needed for your analysis (for example, g has continuous second derivatives).
(c) Estimate the error in $L^2(\Omega)$-norm by using the duality argument.