Numerical Analysis Qualifier: Spring 2000

Do four of the following six problems. **Note that you must choose two from problems 1,2,3 and two from problems 4,5,6. Start each problem on a new piece of paper and write on only one side.**

Problem 1. Let $w(x) > 0$ be continuous on $[a, b]$. Consider the quadrature

$$I(f) = \sum_{i=1}^{n} a_i f(x_i),$$

approximating the integral

$$\int_a^b w(x) f(x)\, dx.$$

A standard theorem shows that there exists nodes $\{x_i\}_{i=1}^n$ and coefficients $\{a_i\}_{i=1}^n$ which make $I(f)$ exact for $\mathcal{P}_{2n-1}$ (the space of polynomials of degree less than or equal to $2n-1$). Show that the nodes $\{x_i\}_{i=1}^n$ and corresponding coefficients $\{a_i\}_{i=1}^n$ are, in fact, unique.

Problem 2. In the system $Ax = b$, the $n \times n$ matrix A is symmetric and nonnegative definite, i.e. $(Ax, x) \geq 0$ for all $x \in R^n$. Here (x, y) denotes the dot product in R^n. Suppose that the only eigenvector corresponding to the zero eigenvalue is the vector $\mathbf{1}$=$(1,\ 1, ...,\ 1)^T$ and the right-hand side b satisfies the condition $(b, \mathbf{1}) = 0$. Consider the iteration

$$x^{(m+1)} = x^{(m)} - \tau(Ax^{(m)} - b).$$

(a) Give conditions on the initial iterate x_0 and iteration parameter τ so that the above iteration converges to the unique solution x of $Ax = b$ satisfying $(x, \mathbf{1}) = 0$.

(b) What is the optimal choice of the iteration parameter τ?

Problem 3. Consider the initial value problem for the ordinary differential equation

$$y' = f(y, t), \quad y\,(t_0) = y_0$$

and a numerical method for approximating its solution of the form

$$u_{n+1} = u_n + h\, M\,(u_n, t_n)\,,$$

where $u_0 = y_0$, $h \in (0, 1)$ is the step size, and $t_{n+1} = t_n + h$.

(a) Define the local truncation error associated with the method and define what it means for the method to be fourth order.

(b) Consider the extrapolated method defined by

$$\begin{aligned} u^1_{n+1/2} &= u_n + \frac{h}{2} f(u_n, t_n), \\ u^1_{n+1} &= u^1_{n+1/2} + \frac{h}{2} f\left(u^1_{n+1/2}, t_n + h/2\right), \\ u^2_{n+1} &= u_n + h\, f\left(u_n, t_n\right), \\ u_{n+1} &= \alpha u^1_{n+1} + \beta u^2_{n+1}. \end{aligned}$$

For what value/values of α and β is the above scheme second order?

Problem 4. Consider the second order hyperbolic problem: Find $u : [0,1] \times [0,T] \to R$ satisfying

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= f \quad \text{in} \quad (0,1) \times (0,T], \\ u(0,t) = u(1,t) &= 0 \quad \text{for} \quad t \in (0,T], \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) &= u_1(x) \quad \text{for} \quad x \in [0,1]. \end{aligned} \tag{1}$$

(a) Let $V_h \subset H^1_0((0,1))$ be a finite element approximation subspace. Define the semi-discrete approximation to (1). You may assume that u_0 and u_1 are functions in V_h.

(b) Consider the following fully discrete time stepping scheme (for $f = 0$): Find $\{W^n\} \subset V_h$ satisfying

$$\begin{aligned} \frac{1}{k^2}(W^{n+1} - 2W^n + W^{n-1}, v) + A(W^n, v) &= 0, \\ \text{for all} \quad v &\in V_h, \\ W^0 = u_0, \quad W^1 &= u_0 + k u_1. \end{aligned} \tag{2}$$

Here $(\cdot,\cdot)$ and $A(\cdot,\cdot)$ denote respectively, the inner product in $L^2((0,1))$ and the Dirichlet form. Let $\{\psi_i\}$ for $i = 1, \ldots, N$ be a complete set of $L^2((0,1))$-orthogonal eigenvectors for the space V_h satisfying the generalized eigenvalue problem

$$A(\psi_i, \theta) = \lambda_i(\psi_i, \theta) \quad \text{for all} \quad \theta \in V_h.$$

Write

$$W^n = \sum_{i=1}^{N} c_i^n \psi_i.$$

Show that for any i, the sequence $\{c_i^n\}$, $n = 0, 1, 2, \dots$ satisfies a three term recurrence.

(c) Using (b), show that (2) is stable if

$$k^2 \max_{i=1,\dots,N} \lambda_i \le 4.$$

Problem 5. Consider the following fourth order boundary value problem: Find $u : [0, 1] \to R$ satisfying

$$\begin{gathered} u'''' = f \quad \text{in} \quad (0,1), \\ u(0) = u''(0) = u'(1) = u'''(1) = 0. \end{gathered} \tag{3}$$

(a) Derive the variational formulation of (3). Identify the variational space V. Which boundary conditions are essential and which are natural?

(b) Show that the quadratic form which appears in the variational formulation is coercive on V.

(c) Define a conforming finite element space for this problem based on a regularly spaced mesh of mesh size $h = 1/N$.

(d) State (do not prove) an error estimate for the finite element method using the approximation space of the previous part.

Problem 6. Consider the boundary value problem for the convection-diffusion equation: Find $u : [0, 1] \to R$ satisfying

$$-\epsilon u'' + cu' = 0, \quad \text{in} \quad (0,1), \quad u(0) = u_0, \ u(1) = u_1,$$

where $\epsilon > 0$, c, u_0, and u_1 are given constants. For solving this problem we consider the following finite difference scheme on a uniform grid $\{x_i = ih : \ i = 0, 1, \dots, n, \ h = 1/n\}$, with $n > 1$ an integer:

$$\begin{gathered} -\epsilon \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + c\frac{y_{i+1} - y_{i-1}}{2h} = 0, \quad i = 1, \dots, n-1, \\ y_0 = u_0, \ y_n = u_1. \end{gathered}$$

Here y_i provides an approximation to $u(x_i)$.

(a) Estimate the local truncation error for sufficiently smooth solutions.

(b) Give conditions on h, c, and ϵ which are sufficient to guarantee that the scheme satisfies the discrete maximum principle. Prove that the discrete maximum principle holds under these conditions.

(c) Give a scheme which is satisfies the discrete maximum principle unconditionally for $c > 0$.