Numerical Analysis Qualifier

January 11, 2000

Complete as many parts of the following problems as you can in the allotted time. In all problems, $\|\cdot\|_m$ denotes the norm in the Sobolev space $H^m(\Omega)$, for $m = 0, 1, \ldots$.

Problem 1. (25 pts.) Let $\Omega = (0,1)^2$. Given $f \in L^2(\Omega)$, consider the following boundary value problem:

$$\begin{cases} -\Delta u &= f, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

(a) Derive a variational formulation for Problem (1). Show that there is a unique solution.

(b) Derive a bound for $\|u\|_1$ in terms of the given data.

(c) For $h \in (0,1]$, let $\mathcal{T}_h = \{\tau_i \ : \ i = 1, \ldots, N_h\}$ be a triangulation of Ω satisfying

$$\max_{\tau \in \mathcal{T}_h} \{\operatorname{diam}(\tau)\} \le h.$$

Define the piecewise linear finite element approximation space S_h for Problem (1) associated with $\mathcal{T}_h$.

(d) State conditions on the collection of triangulations which result in the corresponding approximation spaces satisfying

$$\inf_{\phi \in S_h} \|u - \phi\|_1 \le Ch\|u\|_2,$$

for all $u \in H^2(\Omega) \cap H^1_0(\Omega)$ with $C > 0$ independent of $h \in (0,1]$.

(e) Define the Galerkin approximation of Problem (1) based on S_h. Derive a bound for the H^1-norm of the error for this approximation. You may assume an appropriate Poincaré inequality.

(f) Derive a bound for the error in the $L^2(\Omega)$-norm using the duality argument. You may assume that all solutions of (1) satisfy the *a priori* regularity estimate

$$\|u\|_2 \le c\|f\|_0$$

for some $c > 0$.

Problem 2. (15 pts.) Let A be an $n \times n$ matrix.

(a) Show that if A is diagonally dominant then A is nonsingular.

(b) A matrix A is weakly diagonally dominant if

$$\sum_{j \neq i} |A_{ij}| \leq |A_{ii}|$$

for $i = 1, \dots, n$ with strict inequality for at least one index $i = i_0$. A is irreducible if for each pair $\{i, j\} \subseteq \{1, \dots, n\}$, there is a sequence of pairs

$$(i_1, j_1), \cdots, (i_l, j_l)$$

satisfying

$$\begin{aligned} A_{i_k, j_k} &\neq 0, \text{ for } k = 1, \dots, l, \\ i_1 &= i \text{ and } j_l = j, \\ j_k &= i_{k+1} \text{ for } k = 1, \dots, l-1. \end{aligned}$$

Show that if A is weakly diagonally dominant and irreducible then A is nonsingular.

Problem 3. (20 pts.)

Consider the quadrature of Gaussian type:

$$\int_{-1}^{1} f(x)|x|dx \approx A_0 f(x_0) + A_1 f'(x_1) + A_2 f'(x_2).$$

(a) Show that there is no quadrature of this type that is exact for all polynomials of degree six.

(b) Find nodes and coefficients so that the quadrature is exact for all polynomials of degree less than or equal to five.

Problem 4. (15 pts.)

Consider the initial value problem

$$\frac{dy}{dt} = f(y, t), \quad t > t_0, \quad y(t_0) = y_0,$$

and its approximation by the Runge-Kutta method: For $n = 0, 1, \dots,$

$$u_{n+1} = u_n + h(k_1 + 3k_2)/4, \quad u_0 = y_0, \quad t_{n+1} = t_n + h,$$

where $k_1 = f(u_n, t_n)$ and $k_2 = f(u_n + \frac{2}{3}hk_1, t_n + \frac{2}{3}h)$. Here u_n approximates $y(t_n)$.

(a) Use Taylor's theorem to compute the order of approximation for the method.

(b) Define what it means for $\eta = \lambda h$ to be in the region of absolute stability of the above scheme. Here λ is an arbitrary complex number.

(c) Derive an inequality which can be used to determine when $\eta = \lambda h$ is in the region of absolute stability for this scheme.

Problem 5. (25 pts.) Let $\Omega = (0, 1)$. Consider the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(x,t), \quad x \in \Omega,\ t > 0, \\ u(0,t) = u(1,t) &= 0, \quad t > 0, \\ u(x,0) &= u_0(x), \quad x \in \Omega. \end{aligned} \tag{2}$$

Here $f(x,t)$ is in $L^2(\Omega)$ for each t and $u_0 \in L^2(\Omega)$. Let $S_h \subseteq H_0^1(\Omega)$ be the piecewise linear finite element approximation space based on a mesh of equally spaced nodes with spacing $h = 1/N$. Let $\{\phi_i\}_{i=1}^{N-1}$ be the nodal basis for S_h.

(a) Compute the mass matrix M and stiffness matrix A for S_h.

(b) Consider the following fully discrete approximation of (2):

$$M\left(\frac{U^{n+1} - U^n}{\Delta t}\right) + AU^n = F^n, \quad \text{for } n = 0, 1, \dots . \tag{3}$$

Here, for $i = 1, \dots, N-1$, U_i^n approximates $u(ih, n\Delta t)$ and

$$F_i^n = \int_0^1 f(x, n\Delta t)\phi_i(x)\, dx.$$

Show that (3) is conditionally stable and derive the corresponding CFL condition. (Hint: M and A have a common set of eigenvectors. You may derive the CFL condition in terms of the corresponding eigenvalues. You do not need to compute the eigenvalues explicitly.)

(c) Show that the implicit scheme

$$M\left(\frac{U^{n+1} - U^n}{\Delta t}\right) + AU^{n+1} = F^n, \quad \text{for } n = 0, 1, \dots . \tag{4}$$

is unconditionally stable by providing a bound for $\|U^n\|_0$ for the special case of $F^n = 0$, $n = 0, 1, 2, \dots$.