NUMERICAL ANALYSIS QUALIFYING EXAM

January 1999

Submit any four of the six problems below. Each submitted problem is worth twenty five points.

Problem 1 Consider the linear system of equations $Ax = b$ where A is an $n \times n$ real matrix and $x, b \in R^n$. Assume that $a_{ii} = 1$ for $i = 1, \ldots, n$ and

$$\sum_{j \neq i} |a_{ij}| \leq \gamma |a_{ii}|, \quad \text{for } i = 1, \ldots, n,$$

for some number $\gamma \in [0, 1)$.

(a) Write down the Gauss-Seidel method and define the corresponding reduction matrix G.

(b) Suppose that $y = Gw$ and that i is an index for which $|y_i| = \|y\|_\infty$. Show that

$$\|y\|_\infty \leq \frac{\sum_{j=i+1}^{n} |a_{ij}|}{1 - \sum_{j=1}^{i-1} |a_{ij}|} \|w\|_\infty.$$

In the above expression, we replace the sum in the numerator (resp. denominator) by zero when $i = n$ (resp. $i = 1$).

(c) Use the above inequality to deduce that $\|G\|_\infty \leq \gamma$.

(d) Show that $\|A^{-1}\|_\infty \leq (1 - \gamma)^{-1}$.

(e) Consider using zero as an initial iterate. Using the previous two results, give a bound for the error $\|e_k\|_\infty$ after k Gauss-Seidel steps in terms of $\|b\|_\infty$.

Problem 2 Let $\omega \in C[0, 1]$ with $\omega(x) > 0$ for $x \in [0, 1]$. Consider Gaussian quadrature

$$\int_0^1 \omega(x) f(x)\, dx \cong I_n(f) = \sum_{i=1}^{n} A_i f(x_i).$$

(a) Give expressions for the weights A_i in terms of the nodes $\{x_i\}$.

(b) The nodes $\{x_i\}$ can be computed by finding the roots of a unique monic polynomial of degree n. Describe a procedure for finding this polynomial without using its roots $\{x_i\}$.

(c) Show that $A_i > 0$ for all i.

(d) For $f \in C[0,1]$, show that $I_n(f)$ converges to

$$\int_0^1 \omega(x) f(x)\, dx$$

as n tends to infinity. Hint: Use the fact that the space of polynomials is dense in $C[0,1]$.

Problem 3 Consider the two point boundary value problem

$$\begin{aligned} -u'' &= f \qquad \text{for } x \in (0,1), \\ u(0) &= 0, \;\; u'(1) + u(1) = \mu_1. \end{aligned}$$

(a) Give the weak formulation of the above problem.

(b) Formulate the finite element approximation using piecewise linear functions on a uniform mesh.

(c) Compute the stiffness matrix A.

(d) Show that the spectral condition number of A is bounded from above by Ch^2 where h is the mesh size and C is independent of h.

(e) Show that the spectral condition number of A is bounded from below by ch^2 where c is independent of h.

Problem 4 Let A be a symmetric and positive definite $n \times n$ matrix.

(a) Consider the system of ordinary differential equations for $u(t)$,

$$u_t + Au = 0, \qquad u(0) = u_0. \tag{1}$$

Here $u_0 \in R^n$ and $u(t) \in R^n$ for each $t \geq 0$. Show that $u(t)$ converges to the zero vector as t tends to infinity.

(b) Consider the multistep method

$$\frac{3}{2}x_i - 2x_{i-1} + \frac{1}{2}x_{i-2} = hf(x_i, t_i) \tag{2}$$

for the initial value problem $x' = f(x,t)$. Define what it means for this scheme to be A-stable.

(c) Apply the above scheme to (1) to develop a sequence $\{u_i\}$ where $u_i \in R^n$ approximates $u(ih)$.

(d) Use the fact that (2) is A-stable to show that u_i converges to the zero vector as i tends to infinity.

Problem 5 Consider the following partial differential equation:

$$\begin{aligned} u_t - u_{xx} + 5u_x &= f, \qquad \text{for } x \in (0,1), \\ u(0,t) = u(1,t) &= 0, \qquad \text{for } t \in (0,\infty). \end{aligned}$$

(a) Give an explicit finite difference approximation to the above equation on a uniform grid in both x (space) and t (time) and estimate the local truncation error.

(b) Derive the Courant condition for stability of this scheme.

(c) For the same grid as above, give an implicit scheme which is unconditionally stable in the discrete maximum norm.

(d) For the implicit scheme, consider $f = 0$ and show that the discrete maximum norm of the approximate solution at the $k+1$'st time step is bounded by the discrete maximum norm of the approximate solution at the k'th time step.

Problem 6 Consider the following partial differential equation:

$$\begin{aligned} -\epsilon\Delta u + u_x + u_y + cu &= f, \qquad \text{in } \Omega, \\ u &= g, \qquad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

Here Ω is the unit square $(0,1)^2$, $\partial\Omega$ denotes its boundary, ϵ and c are positive constants and f and g are given functions.

(a) Give the weak form of the above problem.

(b) Let S_h denote the space of piecewise linear functions defined with respect to a triangulation of Ω. Define the corresponding Galerkin approximation to (3).

(c) Define the streamline diffusion approximation to (3) based on S_h.

(d) Give a boundary condition that makes the above problem well-posed when $\epsilon = 0$.