Numerical Analysis Qualifier

January 15, 1998

Attempt any five of the following problems. Each problem is worth 20 points. Indicate the problems to be graded !

Problem 1. Consider the system

$$\begin{pmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

(a) Write down the Gauss-Siedel method for the iterative solution of the above problem.

(b) Give the iteration (or reduction) matrix associated with the above method and compute its spectral radius.

(c) Given that the spectral radius is less that one, provide a bound for the rate of convergence in some norm.

Problem 2. Let $A : R^n \to R^n$ be a symmetric and positive definite linear operator, $B : R^n \to R^m$ be linear and $B^t : R^m \to R^n$ be its transpose. Consider solving the block system

$$M \begin{pmatrix} v \\ p \end{pmatrix} \equiv \begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{1}$$

Here $v, f \in R^n$ and $p, g \in R^m$.

(a) Show that the function p satisfying (1) also satisfies a system of the form

$$Cp = G, \tag{2}$$

where $C = BA^{-1}B^t$. Determine G.

(b) Show that $BA^{-1}B^t$ is positive semi-definite.

(c) Show that (1) is uniquely solvable if and only if C is definite.

(d) Show that for all $q \in R^m$,

$$(Cq) \cdot q = \sup_{\phi \in R^n} \frac{[q \cdot (B\phi)]^2}{(A\phi) \cdot \phi}.$$

In the above equation $v \cdot w$ denotes the usual inner product in both R^n and R^m.

Problem 3. Let $\ldots < x_{-1} < x_0 < x_1 < \ldots$ be given real numbers.

(a) Define the space S of splines of degree k with respect to the above nodes.

(b) Define the B-splines $B_i^k(x)$ of degree $k = 0$ and $k = 1$ with respect to the above nodes.

(c) Suppose that the nodes are equally spaced, i.e. $x_i - x_{i-1} = h$ for $i \in Z$. Let $k \geq 0$ and $f \in C^1(R)$. Show that

$$|f(x) - I^k f(x)| \leq h(k+1)^2 \|f'\|_{L^\infty(R)},$$

where $I^k f(x) = \sum_{i \in Z} f(x_i) B_i^k(x)$.

Problem 4. Consider the Cauchy problem $x' = f(t,x)$, $x(0) = x_0$ and its approximation by the following two-step method:

$$y_0 = x_0 \text{ and } \; y_{i+2} = y_i + 2hf_{i+1}, \; i = 0, 1, ...,$$

where $y_0 = x_0$ and y_1 is given.

(a) Find the order of the method.

(b) Define the stability of a general multi-step method; show that the given above method is stable.

(c) Define A-stability of a general multi-step method; show that the method given above is not A-stable.

Problem 5. Given the boundary value problem

$$-u'' + u' + u = f(x), \; x \in (0,1), \; -u'(0) + u(0) = 1, \; u'(1) = 3.$$

(a) Derive the variational formulation of this problem and describe the energy space.

(b) Let S_h be a finite element space with nodal basis $\{\phi_i\}$. Derive a linear system for computing the nodal values of the Galerkin approximation to the solution.

(c) In the case of piecewise linear finite elements develop a special quadrature formula for the convective term which will lead to a scheme which satisfies the maximum principle.

Problem 6. Consider the space S of bi-quadratics on the unit square $[0,1]^2$, i.e.,

$$S = \{\phi = \sum_{i,j=0}^{2} c_{ij} x^i y^j\}.$$

(a) Show that there is a nodal basis for S with nodes $\zeta_{ij} = (i/2, j/2)$, $i, j = 0, 1, 2$.

(b) For $\phi \in C^0([0,1]^2)$, let

$$F_{ij}(\phi) = \begin{cases} \phi(i/2, j/2) & \text{if } (i,j) \in \{(0,0),(0,2),(2,0),(2,2),(1,1)\}, \\ \int_0^1 \phi(x,0)\,dx & \text{if } (i,j) = (1,0), \\ \int_0^1 \phi(x,1)\,dx & \text{if } (i,j) = (1,2), \\ \int_0^1 \phi(0,y)\,dy & \text{if } (i,j) = (0,1), \\ \int_0^1 \phi(1,y)\,dy & \text{if } (i,j) = (2,1). \end{cases}$$

Show that there exists a basis $\{\psi_{ij}\}$ for S satisfying

$$F_{kl}(\psi_{ij}) = \begin{cases} 1 \text{ if } (i,j) = (k,l), \\ 0 \text{ otherwise.} \end{cases}$$

(c) Let $\tilde{S}$ denote the collection of functions which are bi-quadratic on each square $[i, i+1] \times [j, j+1]$ for each $i, j \in Z$ and continuous on R^2. Let W_{lm}, $l, m \in Z$ be an arbitrary set of nodal values. Show that there exists exactly one function in $W \in \tilde{S}$ satisfying

$$W(l/2, m/2) = W_{lm}, \text{ for all } l, m \in Z.$$

Problem 7. Consider the following problem for the transient heat equation:

$$u_t = u_{xx} - 5u_x + F(x,t), \text{ for } x \in (0,1), \; t > 0,$$

$$u(0,t) = 0, \; u(1,t) = 0, \text{ for } t > 0,$$
$$u(x,0) = f(x), \text{ for } x \in (0,1).$$

(a) Write a finite difference approximation to this problem that is explicit in time and second order in space and estimate the local truncation error.

(b) Using maximum principle type of argument find the exact Courant condition for stability and derive an *á priori* estimate for the solution with respect to the maximum norm.

(c) Give an estimate for the error of the method assuming that the solution $u(x,t)$ is sufficiently smooth.