Numerical Analysis Qualifier
January 1997

Attempt any five of the following problems.

Problem 1 Let A be a symmetric, positive definite (SPD) $n \times n$ matrix and $b \in R^n$. Consider the following iteration

$$x_{k+1} = x_k + M^{-1}(b - Ax_k),$$

where M is a real nonsingular matrix and $x_0 \in R^n$ is a given vector.

(a) Let $x = A^{-1}b$. Show that the error $e_k = x - x_k$ satisfies

$$(Ae_{k+1}, e_{k+1}) = (Ae_k, e_k) - (FM^{-1}Ae_k, M^{-1}Ae_k),$$

where $F = M + M^T - A$ and $(\cdot, \cdot)$ denotes the usual inner product on R^n. From this show that if F is SPD then the iteration $\{x_k\}$ converges to x.

(b) Define the Gauss-Seidel method for solving the linear system $Ax = b$. Now let A be an $n \times n$ tridiagonal matrix defined by

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

Show that the Gauss-Seidel method is convergent and that the error $e_k = x - x_k$ satisfies

$$\|e_{k+1}\|_A \le \left(1 - \frac{C}{n^2}\right) \|e_k\|_A$$

for some positive constant C independent of n. Here $\|\cdot\|_A = (A\cdot, \cdot)^{1/2}$.

Problem 2 Consider the following quadrature rule

$$\int_{-1}^{1} \sqrt{|x|} f(x) dx = A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3).$$

Determine the nodes $x_i (i = 1, 2, 3)$ and the weights $A_i (i = 1, 2, 3)$ such that the resulting quadrature formula is exact for polynomials of degree less than or equal to m with the largest m. Justify your answer.

Problem 3 Let A be SPD on R^n with inner product $(\cdot,\cdot)$ and norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$. Denote by $\rho(A)$ the spectral radius of A. Let

$$W = \{\ v \in R^n :\ \|Av\| = \rho(A)\|v\|\ \}$$

and denote by $W^\perp$ the orthogonal complement of W in R^n. Show that the iteration

$$x_{k+1} = Ax_k/\|Ax_k\|$$

converges to a vector $x \in W$ provided that $x_0 \notin W^\perp$.

Problem 4 Consider the mesh

$$x_i = ih, \quad i = 0, \dots, n, \quad h = \frac{1}{n}$$

on $[0,1]$. Define $\{\psi_j\}_{j=1}^n$ by

$$\psi_j(x) = \begin{cases} 1 & x \in (x_{j-1}, x_j) \\ 0 & \text{otherwise.} \end{cases}$$

Let $S_h = \text{span}\{\psi_j,\ j = 1, \dots, n\}$ and let Q_h be the $L^2(0,1)$ orthogonal projection onto S_h.

(a) Given $f \in L^2(0,1)$ let

$$Q_h f = \sum_{i=1}^n c_i \psi_i.$$

Derive an expression for c_i.

(b) Show that for $f \in C^1[0,1]$

$$\|f - Q_h f\|_{L^\infty(0,1)} \le h\|f'\|_{L^\infty(0,1)}.$$

(c) Show that for $f \in C^1[0,1]$

$$\|f - Q_h f\|_{L^2(0,1)} \le h\|f'\|_{L^2(0,1)}.$$

Problem 5 Consider the following transport equation

$$\begin{array}{ll} u_t + au_x = 0, & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = f(x), & -\infty < x < \infty, \end{array}$$

where a is a positive constant and f is a given smooth function. Consider the grid

$$\Omega_h = \{ \, (x_j, t_n) \equiv (jh, n\Delta t), \quad -\infty < j < \infty, \; n \geq 0\}$$

where h is the space step and Δt is the time step.

(a) Show that the following explicit scheme is absolutely unstable

$$\frac{y_j^{n+1} - y_j^n}{\Delta t} + a\frac{y_{j+1}^n - y_j^n}{h} = 0, \quad j = 0, \pm 1, \ldots, \quad n = 0, 1, \ldots$$

where $y_j^n \cong u(x_j, t_n)$.

(b) Write down an explicit scheme which is conditionally stable.

(c) Consider the same problem over $0 < x < \infty, t > 0$ with a boundary condition $u(0,t) = g(t)$. Write down an implicit scheme which is unconditionally stable.

Hint: You may use the Fourier mode analysis. For parts (b) and (c), you may also use the maximum principle.

Problem 6 Consider the following initial boundary value problem for the 1-D parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + F(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad \dfrac{\partial u}{\partial x}(1,t) + u(1,t) = g(t), & t > 0, \\ u(x,0) = f(x), & 0 < x < 1, \end{cases} \tag{1}$$

where $F(x,t)$, $g(t)$ and $f(x)$ are given smooth functions.

(a) Let S_h be the space of continuous functions which are piecewise linear with respect to the uniform grid partition

$$\Omega_h = \{x_i = ih \mid i = 0, 1, \ldots, N, \; h = 1/N.\}$$

Formulate the semidiscrete Galerkin approximation to eq. (1) using S_h and show that the solution to this problem can be obtained by solving a system of ODEs.

(b) Derive an L^2 norm estimate for the error of the Galerkin solution.

(c) Setup the stiffness matrix of the semidiscrete Galerkin problem.

Problem 7 Let $\Omega \subset R^n$ be a bounded, convex polygonal domain. Let $H = H_0^1(\Omega)$ with inner product and corresponding norm

$$(u, v)_1 = D(u, v) + (u, v) \quad \text{and} \quad \|u\|_1^2 = (u, u)_1,$$

respectively, where

$$(u, v) = \int_\Omega uv dx \quad \text{and} \quad D(u, v) = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx.$$

Define on $H \times H$ the form

$$a_k(u, v) = D(u, v) - k(u, v),$$

where k is a positive constant.

(a) Show that there exists a $k_0 > 0$ such that $a_k(\cdot, \cdot)$ is continuous and coercive on H for $k \leq k_0$.

(b) Let $f \in C^0(\overline{\Omega})$. Show that for $k \leq k_0$ there exists a unique function $u \in H$ such that
$$a_k(u, v) = (f, v), \quad \text{for all} \quad v \in H.$$

(c) Let S_h be a subspace of H and h be a mesh parameter. The Galerkin approximation u_h satisfies

$$a_k(u_h, v) = (f, v), \quad \text{for all} \quad v \in S_h.$$

Assume that S_h has the following approximation property:

$$\inf_{v_h \in S_h} \|v - v_h\|_1 \leq Ch\|v\|_2, \quad \text{for all} \quad v \in H^2(\Omega),$$

where the constant C is independent of h. State the H^1 norm estimate for the error in the Galerkin approximation.

(d) Derive an L^2 norm estimate for the error using the result in part (c).

Hint: use Aubin-Nitsche's duality argument.