Numerical Analysis Qualifier
Fall 1995

Attempt any five of the following problems. Each problem is 20 points. The points in the subproblems indicate the relative weight of that part.

Problem 1. Consider the iteration method

$$x^{(k+1)} = Mx^{(k)} + b,$$

where $x^{(k)}$ and b are vectors in R^n, $M \in R^{n\times n}$, and $x^{(0)}$ is a given initial guess. Assume that $||M|| < 1$, where $||M||$ is a matrix norm subordinate to (induced by) the vector norm $||x||$. Show that:
(a) (5 pts) The process is convergent to the unique solution of the linear system $x = Mx + b$.
(b) (7 pts) Prove that

$$||x^{(k)} - x|| \leq ||(I - M)^{-1}||.||x^{(k+1)} - x^{(k)}||.$$

(c) (8 pts) Show also that

$$||x^{(k)} - x|| \leq ||M||^k ||x^{(0)}|| + \frac{||M||^k.||b||}{1 - ||M||}.$$

Problem 2. Let A be a real $n \times n$ matrix with nonzero diagonal elements.
(a) (3 pts) Formulate the Gauss-Seidel iteration for the system $Ax = b$.
(b) (8 pts) State a necessary and sufficient condition for the convergence of the Gauss-Seidel iteration method. Check whether this condition is satisfied for the matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

(c) (9 pts) Prove that if A is symmetric and positive definite then Gauss-Seidel iteration converges. Hint: show that the matrix norm of the error propagation matrix for Gauss-Seidel that is subordinate to the vector norm $||x||_A^2 = (Ax, x)$ is strictly less than 1.

Problem 3. (a) (10 pts) Let $\langle f, g\rangle = \int_a^b w(x)f(x)g(x)dx$, where $w(x) \geq 0$ is a given weight function on $[a, b]$. Prove that the sequence of polynomials defined below is orthogonal with respect to the inner product $\langle ., .\rangle$:

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x), \quad n > 1,$$

with

$$p_0(x) = 1,\ p_1(x) = x - a_1,\ \ a_n = \langle xp_{n-1}, p_{n-1}\rangle / \langle p_{n-1}, p_{n-1}\rangle, b_n = \langle xp_{n-1}, p_{n-2}\rangle / \langle p_{n-2}, p_{n-2}\rangle.$$

(b) (10 pts) Let $w(x) = 1 - x$ and $a = 0,\ b = 1$. Find the Gaussian quadrature for the integral $\int_0^1 (1-x)f(x)g(x)dx$, which has algebraic degree of accuracy three. Use the general theory by constructing the corresponding orthogonal polynomials.

Problem 4. Given the initial value problem $y' = f(x,y), y(x_0) = y_0$:
(a) (8 pts) Derive a Runge-Kutta method of order two;
(b) (5 pts) Show that the following Runge-Kutta formula is at least of third order:

$$\eta_0 = y_0, \eta_{i+1} = \eta_i + h\Phi(x_i, \eta_i; h), \quad i = 0, 1, ...$$

where

$$\Phi(x, y; h) = \frac{1}{6}(k_1 + 4k_2 + k_3),$$

with

$$k_1 = f(x, y), \quad k_2 = f(x + \frac{1}{2}h, y + \frac{1}{2}hk_1), \quad k_2 = f(x + h, y - hk_1 + 2hk_2).$$

(c) (7 pts) Apply the formulas of (a) and (b) and the automatic step-size control concept to the problem $y' = x + y,\ x > 0,\ y(0) = 1$ to find such a step-size h that guarantees $|\eta_1 - y(h)| \approx 10^{-3}$. Begin with $h = 0.2$.

Problem 5. Consider the boundary value problem:

$$u^{(4)} + q(x)u = 1, \quad 0 < x < 1, \quad u(0) = u'(0) = 0, \quad u'''(1) = 0, \quad u''(1) + u'(1) = 0,$$

where $q(x) \geq 0$ is a given continuous function on the interval $[0, 1]$.
(a) (10 pts) Describe the energy space H_E and give the variational formulation of this problem as a minimization of a quadratic functional over the energy space H_E. Prove that all elements in the energy space H_E satisfy the essential boundary condition $u'(0) = 0$.
(b) (7 pts) Is it true that all elements from the energy space satisfy the boundary condition $u'''(1) = 0$? Justify your answer.
(c) (3 pts) What finite element space can you suggest for minimizing this quadratic functional?

Problem 6. Consider the Crank-Nicolson finite difference scheme for the 1-D heat equation $u_t - u_{xx} = 1,\ \ 0 < x < 1,\ \ t > 0,\ \ u(0,t) = 0,\ \ u_x(1,t) = 1,\ \ t > 0, u(x,0) = 0$;

$$\frac{U_j^{n+1} - U_j^n}{k} = 0.5\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} + 0.5\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} + 1,$$

where k and h are the time and space mesh step sizes, and U_j^n is approximation for $u(x_j, t_n),\ \ x_j = jh,\ \ j = 1, 2, ..., N-1,\ \ Nh = 1,\ \ t_n = nk,\ \ n = 0, 1, ...,\ \ U_j^0 = 0,\ \ U_0^n = 0.$

(a) (6 pts) Estimate the local truncation error of this approximation.
(b) (6 pts) Derive a second order approximation for the boundary condition at $x = 1$ which leads to a symmetric matrix.
(c) (8 pts) Prove that the scheme is unconditionally stable. (You can use Fourier mode analysis, energy type *a priori* estimates or any other method you know.)

Problem 7. Consider the following problem:

$$-\Delta u = -u_{xx} - u_{yy} = 4 \quad in \quad \Omega, \tag{1}$$

$$u = 0 \quad on \quad \Gamma_1, \quad \frac{\partial u}{\partial n} + u = 0 \quad on \quad \Gamma_2. \tag{2}$$

The domain Ω is the triangle with vertices at (0,0), (1,0), and (0,1). The set Γ_1 is the part of the boundary of Ω on the line $x + y = 1$, while Γ_2 is the rest of the boundary. The derivative $\frac{\partial u}{\partial n}$ is in the direction of the exterior normal.
(a) (5 pts) Give the variational formulation of the problem (1) in the form of an integral identity $a(u, v) = (f, v)$ and describe the energy space H_E.
(b) (8 pts) It is desired to find the linear system $K\vec{\mathbf{c}} = \vec{\mathbf{b}}$ if the finite element method is applied to obtain an approximate solution $u_h \in V_h = \{v \in P_2(\Omega) : v \in H_E\}$, and the nodal basis is used. Find the basis functions ϕ_i for V_h if one element is used and the nodes are the vertices and midpoints of the edges of Ω. Also, set up the calculations to find the entries in $K = [k_{ij}]$ and $\vec{\mathbf{b}} = [b_i], \ i, j = 1, 2, 3$.
(c) (7 pts) Calculate k_{12}, given that

$$\int\int_\Omega x^2 = \int\int_\Omega y^2 = \frac{1}{12}, \int\int_\Omega xy = \frac{1}{24}, \int\int_\Omega x = \int\int_\Omega y = \frac{1}{6}.$$

What is the minimal number of distinct inner products $a(\phi_i, \phi_j)$ necessary to calculate the elements of the stiffness matrix K ?