NUMERICAL ANALYSIS QUALIFIER

January 2001

Do four problems, two from Problems 1–3 and two from 4–6. If you attempt more than four problems, please identify the four to be graded.

Problem 1. Let $x_0 < x_1 < \cdots < x_n$ be given real numbers. Let $f(x_0),\ f(x_1), \ldots, f(x_n)$ be the values at the nodes and $f'(x_0)$ and $f'(x_n)$ be the values of the derivative at the end points of a smooth function $f(x)$.

(a) Derive a formula for the algebraic polynomial of the lowest degree that interpolates this data.
(b) Derive an error estimate for the interpolating polynomial assuming that the function $f(x)$ has as many derivatives as needed for your analysis.

Problem 2. Consider the Cauchy problem for the ordinary differential equation:

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = 0.$$

(a) Find the constants a, α and β so that the following implicit Runge-Kutta method is of second order (i.e. has local truncation error $O(h^2)$):

$$\eta_{j+1} = \eta_j + h\Phi(x_j, \eta_j),\ \Phi(x_j, \eta_j) = ak_1,\ k_1 = f(x_j + \alpha h, \eta_j + \beta h k_1).$$

(b) Is this scheme stable? Find the domain of its absolute stability. Is the derived method A-stable?

Problem 3. Consider the linear system of equations $Ax = b$ where $b \in \mathbf{R}^n$ and A is the tridiagonal $n \times n$ matrix given by

$$A = \begin{bmatrix} 2+a & -1 & & & \\ -1 & 2+a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2+a \end{bmatrix}, \quad \text{with } a > 0 \text{ a given constant.}$$

Consider the Richardson iteration method:

$$x_{k+1} = x_k - \tau(Ax_k - b).$$

Here τ is an iteration parameter.

(a) Find the range of the iteration parameter τ in terms of the smallest and largest eigenvalues of the matrix A so that the Richardson iteration is guaranteed to converge for any starting iterate. Find the optimal value for τ which gives rise to the fastest convergence.
(b) Compute bounds for the smallest and largest eigenvalues of A in terms of a. Using these, derive a bound for the reduction factor for the iteration as a function of τ and a. Give a bound for the number of iterations sufficient to reduce the A-norm of the error by a factor of 10^{-6}.

Problem 4. Let $f(t)$ for $t > 0$ and ϕ be in $L^2(\Omega)$. Consider the weak formulation of the heat equation: find $u(t) \in H_0^1(I)$ (for each t) such that

$$(\dot{u}(t), v) + (u_x(t), v_x(t)) = (f(t), v), \quad \text{for all} \quad v \in H_0^1(\Omega),$$
$$u(0) = \phi.$$

Here $(\cdot, \cdot)$ is the inner product in $L^2(I)$, $\dot{u}(t) \equiv \frac{d}{dt}u(t)$ and $u_x \equiv \frac{d}{dx}u$.

(a) Let V_h be the space of conforming linear finite elements with respect to a uniform partition $\mathcal{T}_h$ of I with mesh-size $h = 1/(n+1)$. Define the semi-discrete Galerkin finite element approximation for this problem (denote by $u_h(t) \in V_h$ its solution).

(b) Assume that the standard nodal ("hat") functions $\phi_i(x)$, $i = 1, \ldots, n$ are used as a basis for V_h. Compute the global "mass" and "stiffness" matrices M and A for the semi-discrete Galerkin method written in the form of a system of ODEs:

$$M\dot{U}(t) + AU(t) = F(t).$$

Here $U(t) = (u_h(x_1, t), u_h(x_2, t), \ldots, u(x_n, t))^T$ and $F(t) = ((f, \phi_1), \ldots, (f, \phi_n))^T$ are vectors in $\mathbf{R}^n$ and $x_1, \ldots, x_n$ are the nodes of $\mathcal{T}_h$.

(c) Consider the following discretization for approximating the solution at $t_j = j\Delta t$, $j = 0, 1, \ldots$. For j odd, one applies the explicit Euler method and for j even one applies the implicit Euler method. Show that this explicit-implicit scheme is unconditionally stable.

Problem 5. Let Ω be a polygonal domain in $\mathbf{R}^2$. Let $\mathcal{T}_h$ be a triangulation of Ω of size h and consider the P^1-non-conforming finite element space S_h. This is defined to be the set of functions which are piecewise linear with respect to the triangles of $\mathcal{T}_h$, continuous at the midpoints of the edges of the triangulation and vanish at the midpoints of the edges on the boundary of Ω. This space can be used to approximate the solution of the problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

We define

$$D_h(v, w) = \sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla v \cdot \nabla w \, dx.$$

The approximate solution is the function $U \in S_h$ satisfying

$$D_h(U, \phi) = (f, \phi) \quad \text{for all} \quad \phi \in S_h.$$

Here $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$.

(a) Show that functions in S_h are completely determined by their values at the center of the interior edges (nodes) of the triangulation.

(b) Let $\hat{\tau}$ be the reference triangle with vertices (0,0), (1,0) and (0,1). Compute the (local) stiffness matrix corresponding to the above edge centered nodes.

(c) We define the "jump" $[\phi]$ for $\phi \in S_h$ on each edge e_i in the following way:
(i) If $e_i \subset \partial\Omega$ (i.e. a boundary edge), we let τ_i be the triangle containing e_i and n_i denote the outward normal (from τ_i) on e_i. We define the jump $[\phi]$ to be ϕ restricted to e_i (from τ_i);
(ii) Alternatively, if e_i is an interior edge, we let $\tau_{1,i}$ and $\tau_{2,i}$ denote its neighboring triangles and set n_i to be the outward normal with respect to $\tau_{1,i}$ on e_i. Let $\phi_{l,i}$ be the restriction of ϕ from $\tau_{l,i}$ to e_i. We then define $[\phi] = \phi_{1,i} - \phi_{2,i}$.

Let $E = u - U$ be the error corresponding to the P^1-non-conforming approximation defined above. Show that

$$D_h(E, \phi) = \sum \int_{e_i} \left(\frac{\partial u}{\partial n_i} - c_i \right) [\phi] \, ds.$$

The above sum is over all edges of the triangulation and c_i is any constant (it can be different for each edge e_i).

Problem 6. Consider the integral-differential equation

$$-u'' - Gu = f, \qquad \text{in } (0,1),$$
$$u(0) = 0, \quad u'(1) = 1$$

where

$$(Gu)(y) = \int_0^1 (y - x)^2 u(x) \, dx.$$

(a) Derive a weak formulation for the above problem.
(b) Verify the conditions necessary to apply the Lax-Milgram Theorem and conclude the existence and uniqueness of the weak solution.
(c) Define an appropriate finite element approximation subspace based on piecewise linear functions on a uniform mesh with step-size h and define the Galerkin approximation corresponding to this subspace. State the error estimate in $L^2(\Omega)$ and $H^1(\Omega)$ that one should expect to obtain (in terms of h).