Introduction to the Geometry of Tensors Part 2:

Using geometry in complexity theory

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Strassen's spectacular failure

Standard algorithm for matrix multiplication, row-column:

$$\begin{pmatrix} * & * & * \\ & & \\ & & \end{pmatrix} \begin{pmatrix} * & & \\ * & & \\ * & & \end{pmatrix} = \begin{pmatrix} * & & \\ & & \\ & & \end{pmatrix}$$

uses $O(n^3)$ arithmetic operations.

Strassen (1968) set out to prove this standard algorithm was indeed the best possible.

At least for 2×2 matrices. At least over $\mathbb{F}_2$.

He failed.

Strassen's algorithm

Let A, B be 2×2 matrices $A = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix}$. Set

$$\begin{aligned}
I &= (a_1^1 + a_2^2)(b_1^1 + b_2^2), \\
II &= (a_1^2 + a_2^2)b_1^1, \\
III &= a_1^1(b_2^1 - b_2^2) \\
IV &= a_2^2(-b_1^1 + b_1^2) \\
V &= (a_1^1 + a_2^1)b_2^2 \\
VI &= (-a_1^1 + a_1^2)(b_1^1 + b_2^1), \\
VII &= (a_2^1 - a_2^2)(b_1^2 + b_2^2),
\end{aligned}$$

If $C = AB$, then

$$\begin{aligned}
c_1^1 &= I + IV - V + VII, \\
c_1^2 &= II + IV, \\
c_2^1 &= III + V, \\
c_2^2 &= I + III - II + VI.
\end{aligned}$$

Astounding conjecture

Iterate: $\rightsquigarrow 2^k \times 2^k$ matrices using $7^k \ll 8^k$ multiplications,

and $n \times n$ matrices with $O(n^{2.81})$ arithmetic operations.

Bini 1978, Schönhage 1983, Strassen 1987, Coppersmith-Winograd 1988 $\rightsquigarrow O(n^{2.3755})$ arithmetic operations.

Astounding Conjecture

For all $\epsilon > 0$, $n \times n$ matrices can be multiplied using $O(n^{2+\epsilon})$ arithmetic operations.

$\rightsquigarrow$ *asymptotically, multiplying matrices is nearly as easy as adding them!*

1988-2011 no progress, 2011-14 Stouthers,Williams,LeGall $O(n^{2.373})$ arithmetic operations.

The matrix multiplication tensor

Set $N = n^2$.

Matrix multiplication is a bilinear map

$$M_{\langle n \rangle} : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N,$$

In other words

$$M_{\langle n \rangle} \in \mathbb{C}^{N*} \otimes \mathbb{C}^{N*} \otimes \mathbb{C}^N.$$

Exercise: As a trilinear map, $M_{\langle n \rangle}(X, Y, Z) = \mathrm{trace}(XYZ)$.

Strassen's algorithm as a rank expression

Rank one tensors correspond to bilinear maps that can be computed using one scalar multiplication.

The rank of a tensor T is essentially the number of scalar multiplications needed to compute the corresponding bilinear map.

standard presentation is $M_{\langle n \rangle} = \sum_{i,j,k=1}^{n} x_j^i \otimes y_k^j \otimes z_i^k$

Strassen's presentation is

$$
\begin{aligned}
M_{\langle 2 \rangle} =& x_1^1 \otimes y_1^1 \otimes z_1^1 \\
&+ (-x_2^1 + x_1^2 - x_2^2) \otimes (-y_2^1 + y_1^2 - y_2^2) \otimes (-z_2^1 + z_1^2 - z_2^2) \\
&+ (x_2^1 + x_2^2) \otimes (y_2^1 + y_2^2) \otimes (z_2^1 + z_2^2) \\
&+ (-x_1^2 + x_2^2) \otimes (-y_1^2 + y_2^2) \otimes (-z_1^2 + z_2^2) \\
&+ \mathbb{Z}_3 \cdot [x_2^1 \otimes y_1^2 \otimes (z_1^1 - z_2^1 + z_1^2 - z_2^2)]
\end{aligned}
$$

Tensor formulation of conjecture

Theorem (Strassen): $M_{\langle n \rangle}$ can be computed using $O(n^\tau)$ arithmetic operations $\Leftrightarrow$ $\mathbf{R}(M_{\langle n \rangle}) = O(n^\tau)$

Let $\omega := \inf_\tau \{\mathbf{R}(M_{\langle n \rangle}) = O(n^\tau)\}$

ω is called the *exponent* of matrix multiplication.

Classical: $\omega \leq 3$.

Corollary of Strassen's algorithm: $\omega \leq \log_2(7) \simeq 2.81$.

Astounding Conjecture

$\omega = 2$

Conjecture is about a point (matrix multiplication) lying on a secant r-plane to set of tensors of rank one.

Bini's sleepless nights

Bini-Capovani-Lotti-Romani (1979) investigated if $M_{\langle 2 \rangle}$, with one matrix entry set to zero, could be computed with five multiplications (instead of the naïve 6), i.e., if this reduced matrix multiplication tensor had rank 5.

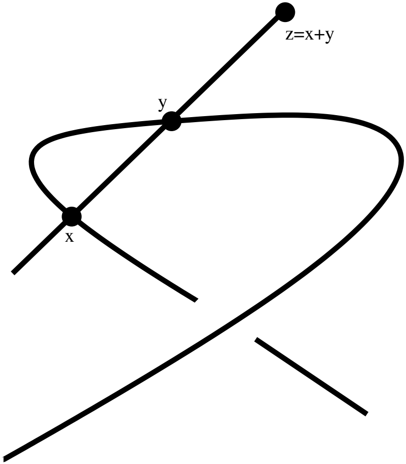
They used numerical methods.

Their code appeared to have a problem.

Recall our picture

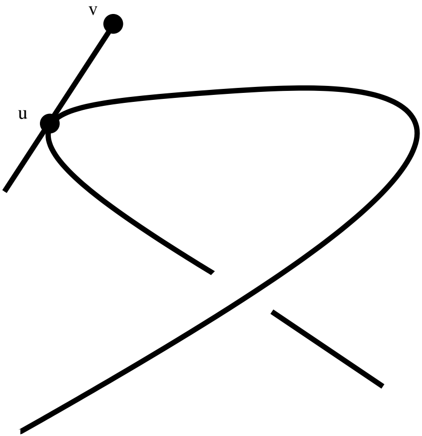
$\{$ tensors of rank two$\}$ $=$

$\{$ points on a secant line to set of tensors of rank one$\}$



Tensors of rank 5: points on a secant 5-plane

Recall our second picture



Theorem (Bini 1980) $\underline{\mathbf{R}}(M_{\langle n \rangle}) = O(n^{\omega})$, so border rank is also a legitimate complexity measure.

Debt from last time

$a(t) \otimes b(t)$ curve of rank one matrices

$a'(0) \otimes b(0) + a(0) \otimes b'(0)$ tangent vector to $a(0) \otimes b(0)$

visibly sum of two rank one elements.

$a(t) \otimes b(t) \otimes c(t)$ curve of rank one tensors

$a'(0) \otimes b(0) \otimes c(0) + a(0) \otimes b'(0) \otimes c(0) + a(0) \otimes b(0) \otimes c'(0)$ tangent vector to $a(0) \otimes b(0) \otimes c(0)$

Exercise: not sum of two rank one elements when $a'(0) \neq \lambda a(0)$, $b'(0) \neq \mu b(0)$, and $c'(0) \neq \nu c(0)$.

How to *disprove* astounding conjecture?

Let $\sigma_r \subset \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N = \mathbb{C}^{N^3}$: tensors of border rank at most r.

Find a polynomial P (in N^3 variables) in the ideal of σ_r, i.e., such that $P(T) = 0$ for all $T \in \sigma_r$.

Show that $P(M_{\langle n \rangle}) \neq 0$.

Embarassing (?): had not been known even for $M_{\langle 2 \rangle}$, i.e., for σ_6 when $N = 4$.

Arora and Barak: lower bounds are "**complexity theory's Waterloo** "

Why I thought this would be easy

Consider rank at most r matrices:
$\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B)) = \{[T] \mid \underline{\mathbf{R}}(T) \leq r\}$

Invariant under changes of bases $\Rightarrow$ its ideal
$I_{\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B))} \subset Sym(A^* \otimes B^*)$ invariant under changes of bases

Special case: rank one - saw matrix has rank one iff size two minors zero. Degree two polynomials.

Consider all homogeneous degree two polynomials on matrices:

$S^2(A^* \otimes B^*) = S^2 A^* \otimes S^2 B^* \oplus \Lambda^2 A^* \otimes \Lambda^2 B^*$

Size two minors ??

Exercise: What about $S^2(A^* \otimes B^* \otimes C^*)$?

Any subspace in $I_{\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))}$?

Why did I think this would be easy?: Representation Theory

Matrices of rank at most r: zero set of size $r+1$ minors.

Tensors of border rank at most 1: zero set of size 2 minors of flattenings tensors to matrices: $A \otimes B \otimes C = (A \otimes B) \otimes C$.

Tensors of border rank at most 2: zero set of degree 3 polynomials.

Representation theory: systematic way to search for polynomials.

2004 L-Manivel: No polynomials in ideal of σ_6 of degree less than 12

2013 Hauenstein-Ikenmeyer-L: No polynomials in ideal of σ_6 of degree less than 19. However there are polynomials of degree 19. Caveat: too complicated to evaluate on $M_{\langle 2 \rangle}$. Good news: easier polynomial of degree 20 (trivial representation) $\rightsquigarrow$

(L 2006, Hauenstein-Ikenmeyer-L 2013) $\underline{\mathbf{R}}(M_{\langle 2 \rangle}) = 7$.

Polynomials via a retreat to linear algebra

$T \in A \otimes B \otimes C = \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ may be recovered from the linear space $T(C^*) \subset A \otimes B$.

tensors up to changes of bases $\sim$ linear subspaces of spaces of matrices up to changes of bases.

Even better than linear maps are endomorphisms. Assume $T(C^*) \subset A \otimes B$ contains an element of full rank. e.g. $T(\gamma_1) : B^* \to A$. Use it to obtain an isomorphism $A \otimes B \simeq \mathrm{End}(A)$ via $T(\gamma)T(\gamma_1)^{-1} : A \to A \rightsquigarrow$ space of endomorphisms.

$\mathbf{R}(T) = N \Leftrightarrow N$-dimensional space of simultaneously diagonalizable endomorphisms (matrices)

$\underline{\mathbf{R}}(T) = N \Leftrightarrow$ limit of N-dimensional spaces of simultaneously diagonalizable endomorphisms

Good News: Classical linear algebra!

Bad News: Open question.

Retreat to linear algebra, cont'd

Simultaneously diagonalizable matrices $\Rightarrow$ commuting matrices

Good news: Easy to Test.

Better news (Strassen): Can upgrade to tests for higher border rank than N: $\underline{\mathbf{R}}(T) \geq N + \frac{1}{2}$(rank of commutator)

$\rightsquigarrow$ (Strassen 1983) $\underline{\mathbf{R}}(M_{\langle n \rangle}) \geq \frac{3}{2}\mathbf{n}^2$

Variant: (Lickteig 1985) $\underline{\mathbf{R}}(M_{\langle n \rangle}) \geq \frac{3}{2}\mathbf{n}^2 + \frac{\mathbf{n}}{2} - 1$

1985-2012: no further progress other than for $M_{\langle 2 \rangle}$.

Retreat to linear algebra, cont'd

Perspective: Strassen mapped space of tensors to space of matrices, found equations by taking minors.

Classical trick in algebraic geometry to find equations via minors.

$\rightsquigarrow$ (L-Ottaviani 2013) $\underline{\mathbf{R}}(M_{\langle n\rangle}) \geq 2\mathbf{n}^2 - \mathbf{n}$

These equations were found using *representation theory*: found via a $G = GL(A) \times GL(B) \times GL(C)$ module map from $A{\otimes}B{\otimes}C$ to a space of matrices (systematic search possible).

Explicitly: $A{\otimes}B{\otimes}C \mapsto \mathrm{Hom}(\Lambda^p A{\otimes}B^*, \Lambda^{p+1}A{\otimes}C)$ Given $T = \sum_{ijk} T^{ijk} a_i{\otimes}b_j{\otimes}c_k$, map is
$a_{s_1} \wedge \cdots a_{s_p}{\otimes}\beta^t \mapsto \sum_{i,k} T^{itk} a_i \wedge a_{s_1} \wedge \cdots a_{s_p}{\otimes}c_k$

Punch line: Found equations by exploiting symmetry of σ_r

Bad News: Barriers

Theorem (Bernardi-Ranestad,Buczynski-Galcazka,Efremenko-Garg-Oliviera-Wigderson): Game (almost) over for determinantal methods.

For the experts: Determinantal methods detect zero dimensional schemes (want zero dimensional smoothable schemes).

Spans of zero dimensional (local) schemes of length $6m$ on Segre fill ambient space. (Bernardi-Ranestad+Buczynski)

In particular, cannot use to show $\underline{\mathbf{R}}(T) > 6m$.

Punch line: **Barrier** to progress.

How to go further?

So far, lower bounds via symmetry of σ_r.

The matrix multiplication tensor also has symmetry:

$T \in A \otimes B \otimes C$, recall the *symmetry group of* T
$G_T := \{g \in GL(A) \times GL(B) \times GL(C) \mid g \cdot T = T\}$

$GL_{\mathbf{n}}^{\times 3} \subset G_{M_{\langle n \rangle}} \subset GL_{\mathbf{n}^2}^{\times 3} = GL(A) \times GL(B) \times GL(C)$:

Proof: $(g_1, g_2, g_3) \in GL_{\mathbf{n}}^{\times 3}$

$$\mathrm{trace}(XYZ) = \mathrm{trace}((g_1 X {g_2}^{-1})(g_2 Y {g_3}^{-1})(g_3 Z {g_1}^{-1}))$$

How to exploit G_T? "Border substitution method"

$\underline{\mathbf{R}}(T) \leq r \Leftrightarrow \exists$ curve $E_t \subset G(r, A{\otimes}B{\otimes}C)$ such that

i) For $t \neq 0$,
$E_t = \mathrm{span}\langle a_1(t){\otimes}b_1(t){\otimes}c_1(t), \ldots, a_r(t){\otimes}b_r(t){\otimes}c_r(t)\rangle$

ii) $T \in E_0$.

For all $g \in G_T$, gE_t also works.

$\rightsquigarrow$ (L-Michalek 2017) can insist on normalized curves, e.g., for $M_{\langle n\rangle}$, E_0 such that $(g_1, g_2, g_3)E_0 = E_0$, when each g_j upper triangular.

$\rightsquigarrow$ $\underline{\mathbf{R}}(M_{\langle n\rangle}) \geq 2\mathbf{n}^2 - \log_2 \mathbf{n} - 1$

$\rightsquigarrow$ Hay in a haystack: A random tensor in $\mathbb{C}^m{\otimes}\mathbb{C}^m{\otimes}\mathbb{C}^m$ has border rank $\sim \frac{m^2}{3}$. Find an *explicit* sequence of tensors of border rank $m^{1+\epsilon}$. previously: $2m - \sqrt{m}$ (L-Ottaviani, 2013) using border substitution: $(2.03)m$ (L-Michalek, 2020).

More bad news: this method cannot go much further.

New idea: Buczynska-Buczynski

Use more algebraic geometry: Consider not just curve of r points, but the curve of **ideals** $I_t \in Sym(A^* \oplus B^* \oplus C^*)$ it gives rise to: *border apolarity method*

$T = \lim_{t\to 0} \sum_{j=1}^{r} a_j(t)\otimes b_j(t)\otimes c_j(t)$

I_t ideal of
$[a_1(t)\otimes b_1(t)\otimes c_1(t)] \cup \cdots \cup [a_r(t)\otimes b_r(t)\otimes c_r(t)] \subset \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C$

Can insist that limiting ideal I_0 is Borel fixed: reduces to small search in each multi-degree.

Instead of single curve $E_t \subset G(r, A\otimes B\otimes C)$ limiting to Borel fixed point, for each (i,j,k) get curve in $G(r, S^iA^*\otimes S^jB^*\otimes S^kC^*)$, each limiting to Borel fixed point *and* satisfying compatibility conditions.

Upshot: algorithm that either produces all normalized candidate I_0's or proves border rank $> r$. $\rightsquigarrow$ significant progress on many border rank problems: see my lecture series

Other Open Problems

• Tensors of minimal border rank: find defining eqns. for $\sigma_m(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}))$ State of art: $m \leq 4$ (Friedland, 2010)

• Even more ambitious: generators of ideal. State of art: $m \leq 3$ (L-Weyman, 2007)

• Cost v. Value in quantum information: Approximate Cost of T $\sim \underline{\mathbf{R}}(T)$, Approximate Value $\sim \underline{\mathbf{Q}}(T)$,

True cost/value $T^{\boxtimes N} := T^{\otimes N} \in (A^{\otimes N}) \otimes (B^{\otimes N}) \otimes (C^{\otimes N})$

$\underset{\sim}{\mathbf{R}}(T) := \lim_{N\to\infty}(\underline{\mathbf{R}}(T^{\otimes N}))^{\frac{1}{N}}$, $\underset{\sim}{\mathbf{Q}}(T) := \lim_{N\to\infty}(\underline{\mathbf{Q}}(T^{\otimes N}))^{\frac{1}{N}}$

Find low cost high value tensors. (see work of Christandl-Vrana-Zuiddam)

Approaches to value

$\mathbf{Q}$, $\underline{\mathbf{Q}}$ not related to classically studied objects.

Idea: define easier to compute quantities bounding $\underline{\mathbf{Q}}$

$\rightsquigarrow$ slice rank (Tao, 2016) and Strength/product rank (for higher order tensors)

Variant over finite fields inspired by random tensors: analytic rank (Gowers) "low (product) rank implies bias" Very recent: Cohen-Moshkovitz: bias implies low (product) rank.

over $\mathbb{C} \rightsquigarrow$ geometric rank (Kopparty-Moshkovitz-Zuiddam, 2020)

$\rightsquigarrow$ classical linear algebra *and* classical algebraic geometry:

spaces of matrices of bounded rank, linear $\mathbb{P}^{m-1}$'s $\subset \mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^m)$ having non-transverse interections with $\sigma_r(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}))$

- Workshop lecture: geometry associated to tensor network states.

Thank you for your attention

For more on **tensors**, their geometry and applications, resp. **geometry and complexity**, resp. **recent developments**: